On Lie algebras associated with normal triality algebras

–In homage of Professor Susumu Okubo (1930-2015)–

– ある代数系の三対原理とリー代数–

Noriaki Kamiya (神谷徳昭)
University of Aizu (福島県公立大学法人会津大学)

概要:　この小論ではリー代数とジョルダン代数を含むある代数系 (called a normal triality algebra) の三対原理，つまり自己同型群と微分の一般化について論究します．そしてその代数系から root systems, Cartan matrix の concept を用いないで例外型の単純リー代数をいくつか構成する方法に応用します. 最後に歴史的な話も述べたいと思います.

§0. Introduction (motivation), §1. Definitions of triality relations and normal triality algebras, §2. Examples of triality relations of algebras, §3. Lie algebras construction, §4. On equivalent's concept of Lie and Jordan algebras, Appendix (対称的合成代数の基底と三対原理 そして　歴史的な事柄), References.

この様な内容について，具体的実例を含め述べさせていただきます (又，これらの section は独立に読める様に心がけたつもりです).

§0. Introduction (motivation)

我々の三対原理の concept を複素数 $\mathbf{C}$ のとき考えます．$\mathbf{C}$ の普通の積を $*$，共役元を $\bar{x} = a - ib(x = a + ib,\ a, b$ は実数,$i = \sqrt{-1})$ とするとき，new product を $xy = \overline{x * y}$ によって定義します．この new product は非結合的代数系 (by $(xy)z \neq x(yz)$) であり, この積で

$$g(xy) = (gx)(gy),\ d(xy) = (dx)y + x(dy)$$

なる自己同型 g と微分 d を考える. $\mathbf{C}$ を $re^{i\theta}$ で表示, 周期 2π で考察すると

$$Aut\ \mathbf{C} = \{e^{i\theta} | \theta = \frac{2\pi}{3}n,\ n : integer\} = \{1, \frac{-1 + \sqrt{3}\ i}{2}, \frac{-1 - \sqrt{3}\ i}{2}\} \cong S_3$$

$$Der\ \mathbf{C} = \{d \in End\ \mathbf{C} | d = \frac{2\pi}{3}in,\ n : integer\} = \{0, \frac{2\pi i}{3}, \frac{4\pi i}{3}\}$$

が成り立ちます. **product (積)** に注意してください.

$j = 0, 1, 2$ に対して (自然数 j を mod 3 で考えます), $End\ \mathbf{C}$ の元で,

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y),\ d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y)$$

となる g_j, d_j を考えると，次のような $Aut\ \mathbf{C}, Der\ \mathbf{C}$ の一般化が得られます. $Trig$, $and\ Trid$ の記号の説明は次の節で行います.

$$Trig\ \mathbf{C} = \{(g_0, g_1, g_2)|\ g_j = exp(\sqrt{-1}\ \alpha_j), \alpha_0 + \alpha_1 + \alpha_2 = 0,\ \alpha_j \in Re\ \mathbf{C}\},$$

$$Trid\ \mathbf{C} = \{(d_0, d_1, d_2)|\ d_j = \sqrt{-1}\ \alpha_j,\ \alpha_0 + \alpha_1 + \alpha_2 = 0,\ \alpha_j \in Re\ \mathbf{C}\}.$$

次に $n \times n$ 次の行列代数 A で考察すると，$j = 0, 1, 2$ に対して

$$\sigma_j(a)x = a_{j+1} * x *^t a_{j+2},\quad d_j(p)x = p_{j+1} * x - x * p_{j+2}.$$

ただし $a_j \in A_0^* := \{b \in A | b *^t b = Id_n\}$, $p_j \in Alt(A) := \{c \in A|\ {}^tc = -c\}$,

($x * y$ は standard product of matrix, ${}^tx$ は transpose of matrix),

とします．ここで $xy = {}^t(x * y)$ と **new product** を定義すると，この積 xy は非結合的代数ですが, 自己同型と微分の一般化が得られます;
$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y), d_j(p)(xy) = (d_{j+1}(p)x)y + x(d_{j+2}(p)y)$, $p_j = (1 - a_j) * (1 + a_j)^{-1}$ (Cayley transformation) が成り立ちます.

Remark. Tits による derivation の一般化の 8 元数の三対原理については ([S],[T],[Tô] and [K.3] 等) を参照.

Remark. Note that for a symmetric composition algebra A (後述), if $\sqrt{3} \in F$ (基礎体), and $dim_F A = 2$, then $AutA \cong S_3$ (3 次の対称群) , if $dim_F A = 1$, then $TrigA \cong K_4$ (Klein's 4 group） ([K-O.3], [K.2]). これらの概念を一般の非結合的代数で以下考えます.

§1. Definitions of triality relations and normal triality algebras

この章で非結合的代数系における一般的な triality relations (local and global の cases) を述べさせていただきます. (a correspondence of local and global triality relations)

Let A be a nonassociative algebra over $ch\ F \neq 2, 3$ (algebra は有限次元そして必ずしも単位元をもたない非結合的代数系の場合を考えます).

Following ([K-O.2] or [K-O.3]), suppose that a triple $g = (g_1, g_2, g_3) \in (Epi\ A)^3$ satisfies a global triality relation

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y) \tag{1.1}$$

where the index j is defined by modulo 3, so that $g_{j\pm 3} = g_j$ (this is said to be a triality group). We denote

$$Trig\ A =$$

$$\{g = (g_1, g_2, g_3) \in (Epi\ A)^3 | g_j(xy) = (g_{j+1}x)(g_{j+2}y), \forall j = 1, 2, 3\} \tag{1.2}$$

This is a generalization of the automorphism group of A, because if $g_j = g,\ j = 0, 1, 2.$, then this g means an automorphism of the algebra A.

In constrast to the algebra triality relations (1.1), we may also consider a local triality relation

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y). \tag{1.3}$$

Analogously to (1.2) if $[t_j, t_k]$ is closed, we introduce

$$Trid\ A =$$

$$\{t = (t_1, t_2, t_3) \in (End\ A)^3 | t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y), \forall j = 1, 2, 3\}. \tag{1.4}$$

Then, it defines a Lie algebra with component wise commutation relation. Also if $(t_1, t_2, t_3) \in Trid\ A$, it is easy to verify that we have for any $\alpha_j \in F, \alpha_{j\pm 3} = \alpha_j$

$$t^{'} = (t^{'}_1, t^{'}_2, t^{'}_3) \in Trid\ A, \text{ where } t^{'}_j = \sum_{k=1}^{3} \alpha_{j-k} t_k \ (j = 1, 2, 3).$$

Furtheremore, if the exponential map $t_j \to \xi_j$ is given by

$$\xi_j = exp\ t_j = \sum_{n=0}^{\infty} \frac{1}{n!} (t_j)^n \tag{1.5}$$

is well-defined, then we can show that

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y), \tag{1.6}$$

provided that $t = (t_1, t_2, t_3) \in Trid\ A$ and vice-versa.

Next we introduce multiplication operators of A, $L(x), R(x) \in End\ A$ by

$$L(x)y = xy \text{ and } R(x)y = yx.$$

Def.1.1. Let $d_j(x, y) \in End\ A$, for $x, y \in A$ $(j = 1, 2, 3)$ be to satisfy

(i)

$$d_1(x, y) = R(y)L(x) - R(x)L(y) \tag{1.7a}$$

$$d_2(x, y) = L(y)R(x) - L(x)R(y). \tag{1.7b}$$

(ii) The explicit form for $d_3(x, y)$ is unspecified except for

$$d_3(x, y) = -d_3(y, x), \tag{1.7c}$$

(iii)

$$(d_1(x, y), d_2(x, y), d_3(x, y)) \in Trid\ A.$$

We call the algebra A satisfying these conditions to be a *regular triality algebra.*

Remark. Any Lie (resp. Jordan) algebra with product $[x, y]$ (resp. xy) is an example of the regular (in particular, normal) triality algebra with respect to the $L([x, y]) = d_j(x, y)$ (resp.$[L(x), L(y)]$) for $j = 1, 2, 3$.

Proposition 1.2 ([K-O.3]). *Let A be a regular triality algebra satisfying either the condition (B) or (C); (B) $AA = A$, (C) if some $b \in A$ satisfies either $L(b) = 0$ or $R(b) = 0$, then $b = 0$. Then we obtain the following.*
(i) For any $t = (t_1, t_2, t_3) \in Trid\ A$, we have

$$[t_j, d_k(x, y)] = d_k(t_{j-k}x, y) + d_k(x, t_{j-k}y) \tag{1.8a}$$

Especially, if we choose $t_j = d_j(x, y)$ it yields

$$[d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y). \tag{1.8b}$$

(ii) For any $g = (g_1, g_2, g_3) \in Trig\ A$, we have

$$g_j d_k(x, y) g_j^{-1} = d_k(g_{j-k}x, y) + d_k(x, g_{j-k}y). \tag{1.8c}$$

Let A be a regular triality algebra with either (B) or (C), and set

$$\mathfrak{D} = span < d_j(x, y), \forall x, y \in A,\ j = 1, 2, 3 > . \tag{1.9}$$

Then $\mathfrak{D}$ is a Lie algebra by (1.8b). Moreover, it is an ideal of the large Lie algebra $Trid\ A$ by (1.8a), denoted by $\mathfrak{D} \lhd Trid\ A$. We call an "inner triality derivation"(naming of the author) this $\mathfrak{D}$ as a generalization of derivations.

Def.1.3. If a regular triality algebra satisfies Eqs.(1.7) as well as

$$d_3(x, y)z + d_3(y, z)x + d_3(z, x)y = 0, \tag{1.10a}$$

$$[d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y), \tag{1.10b}$$

then we call a *pre normal triality algebra* ([K-O.1]). Furthermore, if we have

$$Q(x, y, z) := d_1(z, xy) + d_2(y, zx) + d_3(x, yz) = 0, \tag{1.11}$$

then A is called a *normal triality algebra* ([K-O.2]). Next we introduce the second bilinear product in the same vector space A with involution $x \to \overline{x}$ by

$$x * y = \overline{xy} = \overline{y}\ \overline{x}. \tag{1.12}$$

Then the resulting algebra $(A, x * y)$ is said to be a *conjugation algebra* of A, for the new product $x * y$, by means of $\overline{Qx} = \overline{Q}\ \overline{x}$ and $Q \in End\ A$, we have

$$\overline{g}_j(x * y) = (g_{j+1}x) * (g_{j+2}y),\ \overline{d}_j(x * y) = (d_{j+1}x) * y + x * (d_{j+2}y). \tag{1.13}$$

Remark ([K-O.1]). The conjugation algebra of a structurable algebra which contains an alternative algebra is a normal triality algebra.

Note that the vector space $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ with 182 dimension ([S]) appeared Tits second construction of the Lie algebra of type E_8 is a normal triality algebra (see next section for the details).

Theorem 1.4 ([K-O.2]). *The symmetric composition algebra, Lie and Jordan algebras are a normal triality algebra.*

Theorem 1.5. *For a normal triality algebra A, if we define*

$$\xi_j = exp\ d_j\ (j = 1, 2, 3),\ (\ assuming\ \ the\ well-defined)$$

then we have

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y),\ \text{ that is,}(\xi_j, \xi_{j+1}, \xi_{j+2}) \in Trig\ A,$$

$$[\frac{d}{dt}((exp\ td_j)d_k(exp\ td_j)^{-1})]_{t=0} = [d_j, d_k] \in Trid\ A.$$

That is, this means that $([d_j, d_k], [d_{j+1}, d_{k+1}], [d_{j+2}, d_{k+2}]) \in Trid\ A.$

Corollary. *For the pseudo octonion or para Hurwitz algebras, the same result in Theorem 1.5 holds, as these algebras are a symmetric composition algebra and so a normal triality algebra.*

Remark. In the normal triality algebra A, if we define an endomorphism by $D(x, y) := d_1(x, y) + d_2(x, y) + d_3(x, y)$, then we have the relations $D(x, y) = -D(x, y),\ D(xy, z) + D(yz, x) + D(zx, y) = 0$ and $D(x, y)$ is a derivation satisfying $[D(x, y), D(u, v)] = D(D(x, y)u, v) + D(u, D(x, y)v)$, thus this algebra A is a generalized structurable algebra ([K.1]).

Remark([K-O.1]). The conjugation algebra $(A, x * y)$ of normal triality algebra (A, xy) with a para unit e (i.e., $ex = xe = \overline{x}$) is a structurable algebra with the unit $e * x = x * e = x$, since $x * y = \overline{xy}$ and $\overline{\overline{x}} = x$.

Remark. Let $(A, x * y)$ be an associative algebra. Then $(A, x \cdot y)$ is a Jordan algebra with new product and involution difined by $x \cdot y = x * y + y * x$ and $\overline{x} = x$, since they satisfy the identities $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$. This identity is a definition of the Jordan algebra.

Remark. Let A be a normal triality alg. For $(\xi_1, \xi_2, \xi_3) = (exp\ d_1, exp\ d_2, exp\ d_3) \in exp\ \mathfrak{D}$, provided that the exponential map is well-defined,

$$\forall g = (g_1, g_2, g_3) \in Trig\ A \implies$$

$$g_j\xi_k g_j^{-1} = g_j(exp\ d_k)g_j^{-1} = exp(g_j d_k g_j^{-1}) \in exp\ \mathfrak{D}.\ \text{(by (1.8) and (1.9))}$$

Therefore $G_0 =< \xi_1, \xi_2, \xi_3 >_{span}$ is an invariant subgroup of $Trig\ A$. We call an "inner triality group"(naming of the author) and we obtain $\mathfrak{D} \longleftrightarrow G_0$.

For the definitions of this section, we would like to refer ([K-O.1],[K-O.2]and [K-O.3]), that is, for the concept of normal triality algebras and related topics. It seems that this concept (called a normal triality algebra) of a generalization of the derivation may be regarded as a generalization in the "*principle of triality*" of the octonion (or para octonion) algebra due to Tits ([K.3]).

§2. Examples of triality relations of algebras

For several examples of normal triality algebras, in particular, to construct Lie algebras, we will exhibit them in this section.

First, it is known that a *symmetric composition algebra* A over a field $F(ch\ F \neq 2)$ with a symmetric bilinear form $< x, y >$ satisfying the relations

$$(xy)x = x(yx) =< x, x > y, \ < x, y > \ \ is\ non-degenerate,$$

is a normal triality algebra and the symmetric composition algebra is either a *para-Hurwitz algebra* (that is, the conjugation algebra is a Hurwitz algebra which contains the Cayley number if $ch\ F = 0$), or an *eight dimensional pseudo octonion algebra* due to M.Gell-mann ([G],[K-O.2],[K-O.3], [K3] and [O]). Note that this symmetric algebra satisfies the relation $< xy, xy >=< x, x >< y, y >$, i.e., $||xy|| = ||x||||y||$, where $< x, x >= ||x||^2$.

We denote the conjugation algebra of the Cayley algebra by $\mathbf{O}$ and the pseudo octonion algebra by $\mathbf{O_p}$, this $\mathbf{O_p}$ is called an *Okubo algebra.*

This symmetric composition algebra A has a triality derivation $(d_0, d_1, d_2) \in Trid\ A$ defined by $d_0(x, y) = 2\{[L(x), L(y)] - R([x, y])\}$,

$$d_1(x, y) = R(y)L(x)-R(x)L(y),\ and\ d_2(x, y) = L(y)R(x)-L(x)R(y). \quad (\spadesuit)$$

The case of the symmetric composition algebra;

$$< d_j(a, b) >_{span} \cong D_4\ (Lie\ algebra\ of\ 28\ dimension),$$

since $< d_j(a, b)x, y > + < x, d_j(a, b)y >= 0,\ (\forall\ j = 0, 1, 2)$.

Secondarily, the Lie algebra ($\varepsilon = -1$) and Jordan algebra ($\varepsilon = 1$) are a normal triality algebra equipped with the product $L(x)y = xy = \varepsilon yx = \varepsilon R(x)y$ and the triality derivation $d(x, y) := d_0(x, y) = d_1(x, y) = d_2(x, y) = -\varepsilon[L(x), L(y)]$.

The other examples are the following:

Example a) ([K-O.2]). The vector space $\mathbf{O} \otimes \mathbf{O}$ with 64 dimension induced from two para octonion algebras $\mathbf{O}$ (para octonion is the conjucation algebra of octonion $\mathbf{O}$) is a normal triality algebra with respect to the $D_j(a \otimes x, b \otimes y) := d_j^{(1)}(a, b)\otimes < x, y > id+ < a, b > id \otimes d_j^{(2)}(x, y)$, where $d_j^{(1)}$ and $d_j^{(2)}$ with the triality derivation defined by ($\spadesuit$) respectively. Note that $D = D_0 + D_1 + D_2$ construct a Lie algebra of type $G_2 \oplus G_2$. This implies that $Der\ O \cong G_2$(Lie algebra of 14 dim) in particular.

As we will show in next section, this case is relevant for a construction of so-called Freudenthal's magic square.

Remark. A vector of the tensor product $\mathbf{O_p} \otimes \mathbf{O_p}$ induced from the pseudo octonion algebra $\mathbf{O_p}$ is a normal triality algebra with $D_j(a \otimes x, b \otimes y) = d_j^{(1)}(a,b) \otimes < x,y > id + < a,b > id \otimes d_j^{(2)}(x,y)$ as well as the tensor product of the para octonion algebra O. However note that $D = D_0 + D_1 + D_2$ construct a Lie algebra of type $A_2 \oplus A_2$. Hence this means that $Der\ O_p \cong A_2$ (Lie algebra of 8 dim).

Example b) ([K-O.1], [K-O.3]). For the octonion algebra **O** and a para Zorn vector matrix with 56 dimension induced from the vector space

$$\begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \qquad (\diamondsuit)$$

is a normal triality algebra, where $\forall\ \mathbf{a}, \mathbf{b} \in H_3(\mathbf{O}) (= \mathfrak{J})$ is a exceptional Jordan algebra of degree three with 27 dimension over a field F, and $\alpha, \beta \in F$ (the base field). For the details, see section 5 in ([K-O.3]). This concept was appeared a meta symplectic geometry due to H. Freudenthal (for example, see ([K.5], [K-O.4])).

Example c). The vector space $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ with 182 dimension appeared Tits second construction of E_8([S],[Tô]) is a normal triality algebra with the (triality) derivation $D_0 = D_1 = D_2 (= D)$. Here this derivation of $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ means that $D = (d_0 + d_1 + d_2) \otimes < x,y > id + < a,b > id \otimes [R(x), R(y)]$, where d_j is the triality derivation of the normal triality algebra $\mathfrak{A}$ (as the para octonion algebra) and $[R(x), R(y)]$ is the derivation of the normal triality algebra $\mathfrak{J}$ (as the exceptional Jordan algebra with $dim\ \mathfrak{J} = 27$). Here denote by $\mathfrak{A}_0 = \{a \in \mathfrak{A} | trace\ (a) = 0\}$, $\mathfrak{J}_0 = \{x \in \mathfrak{J} | Trace\ (x) = 0\}$, and $dim\ \mathfrak{A}_0 = 7$, $dim\ \mathfrak{J}_0 = 26$ respectively.

§3. Lie algebras construction

In this section, we will discuss a construction of Lie algebras associated with the normal triality algebras of Examples a), b) and c) in section 2.

Following (]K-O.2]), let A be a normal triality algebra and consider linear maps:

$$\rho_j : A \to V, \text{and } T_j : A \otimes A \to V \qquad (3.1)$$

for $j = 0,1,2$, where V is an unspecified algebras with skew symmetric bi-linear product $[\circ, \circ]$. We set now

$$T(A,A) = span < T_j(x,y), \forall x,y \in A >, \forall j = 0,1,2. \qquad (3.2)$$

$$L(A) = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T(A,A). \tag{3.3}$$

Let (i,j,k) be a cyclic permutation of indices $(0,1,2)$, and assume the following anti-communtative multiplication relations:

$$[\rho_i(x), \rho_i(y)] = -[\rho_i(y), \rho_i(x)] = T_{3-i}(x,y) \tag{3.4a}$$

$$[\rho_i(x), \rho_j(y)] = -[\rho_j(y), \rho_j(x)] = -\rho_k(xy) \tag{3.4b}$$

$$[T_l(x,y), \rho_j(z)] = -[\rho_j(z), T_l(x,y)] = \rho_j(d_{l+j}(x,y)z) \tag{3.4c}$$

$$\begin{aligned}[T_l(u,v), T_m(x,y)] &= T_m(d_{l-m}(u,v)x, y) + T_m(x, d_{l-m}(u,v)y) \\ &= -T_l(d_{m-l}(x,y)u, v) - T_l(u, d_{m-l}(x,y)v)\end{aligned} \tag{3.4d}$$

for $l, m = 0,1,2$. Hence, we introdcuce the Jacobian in $L(A)$ by

$$J(X,Y,Z) = [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \tag{3.5}$$

for $X, Y, Z \in L(A)$.

Lemma 3.1. *$T(A,A)$ and $T_j(A,A)$ for $j = 0,1,2$ are Lie algebras. Also $T_j(A,A)$ is an ideal of $T(A,A)$, where $T_j(A,A) = span < T_j(x,y), \forall x, y \in A >$.*

Condition (D) Suppose that we have $\rho_j(x) = 0$ for some $x \in A$ and for some value of $j = 0,1,2$, we then have $x = 0$.

Proposition 3.2. *Let A be a pre-normal triality algebra. If we have*

$$J(x,y,z) = T_0(x,yz) + T_1(z,xy) + T_2(y,zx) = 0, \tag{3.6}$$

then $L(A)$ is a Lie algebra. Moreover, if the condition (D) holds, then A is a normal triality algebra. Conversly, if $L(A)$ is a Lie algebra and if the condtion (D) holds, then A is a normal triality algebra with the validity of Eq.(3.6).

Theorem 3.3. *Let A be a normal triality algebra. Then, the quotient algebra $\tilde{L}(A) = L(A)/J$ is a Lie algebra, where $J = span < J(x,y,z) >$.*

この定理 3.3 を適用して 5-graded $g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ の例外型リー代数を構成します. 特に normal triality algebra A の inner triality derivation d_j を T_j とみなす.

$$L(A) = \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A) \oplus T(A,A), \ \textit{and}$$

$$T(A,A) = < T_j(a,b), \forall j = 0,1,2, \forall a, b \in A >_{span},$$

and we identify $T_j(a,b)$ with the triple $T_j(a,b) = (d_j(a,b), d_{j+1}(a,b), d_{j+2}(a,b))$.

§2 の Example a),b),c) の場合に付随した Lie algebras を考察する. 以後, 基礎体 F は標数 0 の代数閉体とする.

a) $A = \mathbf{O} \otimes \mathbf{O}$ の場合 (tensor product case , and $dim\ g_{-2} = dim\ g_2 = 14$): $A = A_1 \otimes A_2,\ dim\ A_1, dim\ A_2$ なる記号を用いると, それぞれ construct される Lie algebra は以下の様になります, ただし $\mathbf{O}$ は octonion algebra.

$$L(A) \cong E_8,\ L_j = \rho_j(A) \oplus T_{3-j}(A,A) \cong D_8, T_j(A,A) \cong D_4 \oplus D_4.$$

$dimA_2 \backslash dimA_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

E_8 を Extended Dynkin diagram で表すと $L(A)/L_j(A)$ は 128 次元の対称空間;

◎ − ○ − ○ − ○ − ○ − ○ − ○−$\boxed{\circ}$ $\boxed{\circ}$ omitted ≅ D_8, and ◎ is hieghest root.

|

○

b) $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ の場合 (balanced case, and $dim\ g_{-2} = dim\ g_2 = 1$):

$L(A) \cong E_8,\ L_j = \rho_j(A) \oplus T_{3-j} \cong E_7 \oplus A_1,\ T_j(A,A) \cong E_6 \oplus gl(1) \oplus gl(1)$. ここで $H_3(\mathbf{O}) \rightarrow H_3(\mathfrak{A})(= B)$ なる記号に置き換える. ただし $\mathfrak{A}$ は Hurwitz algebras over F. $\forall \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} F & B \\ B & F \end{pmatrix} = A$ とすると $dimB$ に対応する Lie algebra は以下の様になります.

$dimB$	1	6	9	15	27
$dimA$	4	14	20	32	56
$dimL(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

この場合の construct された E_8 を Extended Dynkin diagram で表すと $L(A)/L_j(A)$ は 112 次元の対称空間;

◎− $\boxed{\circ}$− ○ − ○ − ○ − ○ − ○ −○ $\boxed{\circ}$ omitted ≅ $A_1 \oplus E_7$.

|

○

c) $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$ の場合: Here denote $\mathfrak{A}_0 = \{x \in \mathbf{O} | trace\ x = 0\}, \mathfrak{J}_0 = \{x \in H_3(\mathbf{O}) | Trace\ x = 0\}$, $(dim\ A = 7 \times 26,\ dim\ g_0 = 66)$:

$L(A) = Der(A) \oplus A \cong E_8,\ Der(A) = T(A,A) = Der\mathfrak{A} \oplus Der\mathfrak{J} \cong G_2 \oplus F_4 =< D(X,Y) >_{span}$. この場合の $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ は $X \circ Y = (a * b) \otimes (x * y)$ and $[X,Y] = D(X,Y) + X \circ Y$ による generalized structurable algebra からのリー

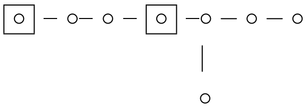
代数の構成です ([K.1]). b) の場合と同様に $\mathfrak{J} = H_3(\mathbf{O}) \to H_3(\mathfrak{A})\ (= B)$ と置く. 詳しい説明は省きますが 248 次元の 5-graded Lie algebra's construction;

$$L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2,\ and\ g_0 = Der\ (g_{-2} \oplus g_{-1} \oplus g_1 \oplus g_2)$$

with respect to the $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$ with the certain algebraic structure.

	dim B=1	dim B=6	dim B=9	dim B=15	dim B=27
$dim\mathfrak{A} = 1$	0	A_1	A_2	C_3	F_4
$dim\mathfrak{A} = 2$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$dim\mathfrak{A} = 4$	A_1	C_3	A_5	D_6	E_7
$dim\mathfrak{A} = 8$	G_2	F_4	E_6	E_7	E_8

この場合の construct された E_8 を Dynkin diagram で表すと, $L(A)/(G_2 \oplus F_4)$ は 182 次元の reductive homogeneous space であり, 次のように予想されます.



$\boxed{\circ}$ omitted $\cong G_2 \oplus F_4 = Der\ A.$

ここでは　Examples a), b) and c) の (extended) Dynkin diagram に関して, E_8 についてだけ具体的に述べましたが, G_2, F_4, E_6, E_7 の場合も同様に表せます.

For the triality relations of Lie algebras associated with triple systems (or structurable algebras) without using root systems, to see our references.

つまり, 上記の normal triality (super)algebras の場合と三項系代数からのリー代数 (リー超代数) の構成が存在します. 三項系 (Freudenthal-Kantor triple systems) からの構成に関し, super 化を含め ([K.5], [K-O.1]～[K-O.4]) など筆者の文献とそこで引用しました著作を参照してください.

以上の結果は $g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ の 5-graded リー（超) 代数の構成には Jordan and alternative algebras を含む normal triality algebras とその super 化の概念が重要であることを示しています.

つまり要約すると, 何故この様な概念を考察するのかの理由をこの章の最後に述べます.

$\cdot\ E_8/D_8$ (182 dim), $E_8/(A_1 \oplus E_7)$ (112 dim) $\cdots$ *symmetric space*

$\cdot\ E_8/(G_2 \oplus F_4)$ (182 dim), $\cdots$ *reductive homogeneous spase*

$\cdot\ E_7/(A_1 \oplus D_6)$ (64 dim), $E_6/(A_1 \oplus A_5)$ (40 dim), $F_4/(A_1 \oplus C_3)$ (28 dim), $G_2/(A_1 \oplus A_1)$ (8 dim), $\cdots$ *quaternionic symmetric space*

$\cdot C(n+1)/(gl(1) \oplus C_n)$ ($4n$ dim), $B(0,n)/C_n$ ($2n$ dim), $G(3)/(A_1 \oplus G_2)$ (14 dim), $F(4)/(A_1 \oplus B_3)$ (16 dim), $\cdots$ *super symmetric space*

このような幾何学の様相が三項系又は normal triality algebra から構成可能だからです.

geometry concept ⇄ algebraic characterization (nonassociative algebras)

§4. On equivalent's concept of Lie and Jordan algebras

この章ではリー代数とジョルダン代数が対等の概念ではないかという筆者の関係する研究分野の非結合的代数系の立場から (MSC2020 の分類), この機会に少しだけ述べたいと思います. ご存知の事柄が多々存在するかもしれませんが今後の課題 (何故　リー三項系, ジョルダン代数系等, 三項系が日本においてはあまり研究されていないのかの理由を含めまして) への一つの提案・歴史の所説として提供させていただきます.

Math. Subject Classification　(17-XX, nonassociative algebras);
17A(1973-now) general nonassociative rings,
17B(1973-now) Lie algebras and Lie superalgebras,
17C(1973-now) Jordan algebras (algebras, triple systems and pairs),
17D(1980-now) other nonassociative rings and algebras,
もう少し詳しく述べますと以下の通りです.
17A40(1980-now) ternary composition,
17A70(1980-now) superalgebras,
17B81　Applications of Lie (super)algebras
17C37(1991-now) associated geometries of Jordan algebras,
17C70(1991-now) super structures,
17C90(1991-now) Applications of Jordan algebras to physics,
17D05(1973-now) Alternative rings.

以上の分類から見られますように,　リー代数 (単位元を持たない) とジョルダン代数 (単位元を持つ) は非結合的代数系の中で考察すると 17B, 17C という様に大きく 2 つに分類されています. ここでジョルダン代数の定義を与えておきます.　体 F 上のベクトル空間 V で 2 項積 xy と双線形性をもち, 次の条件を満たす代数系をジョルダン代数 (Jordan algebra) と言う.

1) $xy = yx,\ \forall x, y \in V,$
2) $(x^2y)x = x^2(yx),\ \forall x, y \in V.$

実例. 行列代数 $(V,\ x * y)$ で新しい積を $xy = x * y + y * x$ で定義するとジョルダン代数です, また 4 元数代数 $(\mathbf{H}, *)$, 8 元数代数 $(\mathbf{O}, *)$ もこの積 xy でジョルダン代数です.

Lie algebras and Jordan algebras のネーミングについて記録として（部分的な抜粋です) 以下の事柄を述べておきます;

(*)The achievement of Lie had been to reduce the study of local properties of an analytic group to the study of corresponding properties of a certain non-associative algebra called its infinitesimal group. In his lecture, Weyl proposed an independent study of such algebras, which became known as Lie algebras (N. Jacobson の personal history より).

(*)The study of such algebras seems to have been initiated by P. Jordan in connection with quantum mechanics and we shall call them *Jordan algebras*

(A.A. Albert の論文より).

体 F 上のベクトル空間 V で 2 項積 xy で閉じていて双線形性をもち, 次の条件を満たす代数系を交代代数 (alternative algebra) といいます.

1) $x^2y = x(xy),\ \forall x, y \in V$,

2) $yx^2 = (yx)x,\ \forall x, y \in V$.

この代数系に関しては次の Artin's theorem が知られています.

The subalgebra generated by any two elements of an alternative algebra V is associative.

一方, 交代代数 (V, xy) で $x \circ y = xy + yx$ と新しい積 $\circ$ を定義するとジョルダン代数です. 更に, 行列代数 (V, xy) で考察すると次のことも成り立ちます.

$[x, y] = xy - yx \Longrightarrow$ Lie algebra,

$x \circ y = xy + yx \Longrightarrow$ Jordan algebra.

まとめると歴史的にはジョルダン代数は 1934 年の Jordan-Wigner-Neumann の論文で単純なものが分類されて (量子力学の特徴ずけの為, 最初のころは r-number systems と呼ばれ上記のように A.A.Albert により Jordan algebra と naming された), リー代数と名ずけられた論文は 1935 年の Jacobson の論文が最初と思われますが Lie algebras と Jordan algebras は独立に, 別々に発展し, 1950 年代項から, Chevalley-Schafer の F_4, E_6 の構成に, Freudenthal の 56 次元 metasymplectic geometry の概念にそれぞれ 27 次元のジョルダン代数 $H_3(\mathbf{O})$ が表れて, その後 Loos, Neher によるジョルダン三項系と幾何学へと研究が継続されていると思います. 筆者は Jordan triple system をもう少し一般化した代数系でリー (超) 代数を含めた construction を考察しています. 代数系から幾何学へ; A generalization of Jordan algebra $\Longrightarrow$ symmetric space, reductive homogeneous space, bounded symmetric space (非結合的代数系の幾何学への応用）.　例えば　佐武氏の　Algebraic structures of symmetric domains (1980). 一方, triple system の物理への応用に関しては quark theory, field theory に関連する論文が存在します. そして大久保氏の Introduction to octonion and other nonassociative algebras in physics(1995) も存在します.

何故　リー（超）代数とジョルダン（超）代数が対等でないのかという疑問には答えがまだないのですがジョルダン代数の表現論の形成が完全には存在しないので, またリー群とリー代数の対応の様にジョルダン代数と群（自己同型群を含む）の作用等, まだまだ未解決の分野が多く存在する事実等と関係するのかもしれません. そして物理学との関連を含めて Zelmanov(Fields 賞受賞者）達が研究しています Jordan superalgebras がこの分野の開拓に役立つのかもしれません.

ーー nonassociative algebras における Fanky 星人（Jack 星人の同胞）が多数現われることを期待してーー

Appendix I (対称的合成代数の基底と三対原理).

Let A be a symmetric composition algebra over the field F and $\sqrt{3} \in F$.

$$\Sigma = \{a = (a_1, a_2, a_3) \in A^3 | a_j a_{j+1} = a_{j+2},\ < a_j, a_j >= 1,\ \forall j = 1, 2, 3\}. \tag{A.1}$$

For a given $a = (a_1, a_2, a_3) \in \Sigma$, we introduce a notation

$$\Lambda(a) = \{p = (p_1, p_2, p_3) \in A^3 |$$

$$a_j p_{j+1} + p_j a_{j+1} = p_{j+2}, < p_j, a_j >= 0,\ \forall j = 1, 2, 3\} \tag{A.2}$$

Note that $\Lambda(a)$ is a vector space over F.

Moreover, we define $q_j \in A$ by $q_j = a_{j+1} p_{j+2} = p_j - p_{j+1} a_{j+2}$.

From (Th.2.5 and Th.3.2 in [K-O.3]) and the notation being as avobe, we have the following theorems for global and local triality relations.

Theorem A.1. *For* $\forall a, b \in \Sigma$, *we have* $\sigma_j(a)$ *and* $\theta_j(a) \in Trig\ A$, $G =< \theta_j(a)\sigma_j(b)$ *and* $\sigma_j(a)\theta_j(b) >_{span}$ *is an invariant subgroup of* $TrigA$, $\sigma_{j+2}(a)\sigma_{j+1}(a)\sigma_j(a) = Id$, $\theta_{j+2}(a)\theta_{j+1}(a)\theta_j(a) = Id$, *(inner triality group) where* $\sigma_j(a) = R(a_{j+1})R(a_{j+2})$ *and* $\theta_j(a) = L(a_{j+2})L(a_{j+1})$.

Theorem A.2. *For any* $a \in \Sigma$ *and* $p \in \Lambda(a)$, *if we introduce* $D_j(a, p) \in End\ A$ *by* $D_j(a, p)x = (p_{j+1}x)a_{j+1} + a_j(xq_j)$, *then this* $D_j(a, p)$*satisfies*

$$D_j(a, p)(xy) = (D_{j+1}(a, p)x)y + x(D_{j+2}(a, p)y)\ (\textit{inner triality derivation}).$$

Furtheremore, we obtain that $T_j(A, A) =< D_j(a, p) >_{span}$ *is an ideal of* $Trid\ A$.

また 8 次元の symmetric composision algebra の場合, u, v を適当に選ぶと

$D_j(a, p) < ----- > d_j(u, v)$ (normal triality algebra における local triality derivation) の対応が知られています ([K-O.3]). Hence, $< d_j(u, v) >_{span} \cong D_4$ (28 次元のリー代数) なので $D_j(a, p)$ を構成する 28 個の $a = (a_1, a_2, a_3)$ *and* $p = (p_1, p_2, p_3)$ を具体的に以下列挙させていただきます. これらの基底に関する乗積表は para and pseudo octonion algebras はそれぞれ次の表で, 内積は $x = \sum_j x_j e_j,\ y = \sum_j y_j e_j$ のとき $< x, y >= \sum_j x_j y_j$ で定義されます.

$(*)$ para octonion の基底: $e_1, \cdots, e_7$ の積の例, $e_1 e_2 = -e_3, e_7 e_1 = -e_6, e_6 e_7 = -e_1$.

para case	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	$-e_3$	e_2	$-e_5$	e_4	$-e_7$	e_6
e_2	e_3	-1	$-e_1$	e_6	$-e_7$	$-e_4$	e_5
e_3	$-e_2$	e_1	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	e_5	$-e_6$	e_7	-1	$-e_1$	e_2	$-e_3$
e_5	$-e_4$	e_7	e_6	e_1	-1	$-e_3$	$-e_2$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	$-e_5$	$-e_4$	e_3	e_2	e_1	-1

(AI) Examples of triality group and derivation in the para octonion algebra (the conjugation algebra of Cayley algebra), $e_0 = 1$ なので省略. $a = (a_1, a_2, a_3), p = [p_1, p_2, p_3]$ と表示する.

(AI):(para octonion algebra), with respect to basis $e_1, \cdots, e_7$.

(1) $a = (e_1, e_2, -e_3),\ p = [e_2, -e_1, 0]$,

Remark. $a = (a_1, a_2, a_3) \in \Sigma$ に対して $p_j = a_{j+1}p_{j+2} + p_{j+1}a_{j+2}$ を満たす $p = [p_1, p_2, p_3]$ の選び方はいくつか存在する. 例えば, $p = [e_3, 0, -e_1]$. $p = [0, e_3, -e_3]$ があります. (2)$(e_1, e_4, -e_5), [e_4, -e_1, 0]$ (3)$(e_1, e_6, -e_7), [e_6, -e_1, 0]$ (4)$(e_2, e_4, e_6), [e_4, -e_2, 0]$ (5)$(e_2, e_5, -e_7), [e_5 - e_2, 0]$ (6)$(e_3, e_4, -e_7), [e_4, -e_3, 0]$ (7)$(e_3, e_5, -e_6), [e_5, -e_3, 0]$ (8)$(\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2, -1,), [\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2, \frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2, 0]$, この (8) の $(i, j) = (1, 2)$ を除いて (9)〜(28): $i \neq j, (\frac{1}{2}e_i + \frac{\sqrt{3}}{2}e_j, \frac{1}{2}e_i + \frac{\sqrt{3}}{2}e_j, -1), [\frac{\sqrt{3}}{2}e_i - \frac{1}{2}e_j, \frac{\sqrt{3}}{2}e_i - \frac{1}{2}e_j, 0]$ つまり $(\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7, \frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7, -1), [\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7, \frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7, 0]$ etc.　そして (8)〜(28): ${}_7C_2 = 21$ 通り存在します. where $e_0e_i = e_ie_0 = \overline{e}_i = -e_i\ (i \neq 0), e_0$ *is a para unit.*

Note that the product $[e_j, e_k] := e_je_k - e_ke_j,\ (j,\ k = 1, \cdots, 7)$ makes a Malcev algebra with seven dimension ([O]). 勿論 $(e_ie_j)e_i = e_j$ は満たします.
(∗∗) pseudo octonion の基底: $e_1, \cdots, e_8$ の積の例, $e_1e_2 = e_3,\ e_8e_1 = e_1, e_8e_8 = -e_8$.

pseudo case	e_1	e_2	e_3	e_4
e_1	e_8	e_3	$-e_2$	$\frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7$
e_2	$-e_3$	e_8	e_1	$\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$
e_3	e_2	$-e_1$	e_8	$\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5$
e_4	$\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$-\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	$\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$
e_5	$\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{1}{2}e_3 - \frac{\sqrt{3}}{2}e_8$
e_6	$\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	$\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7$	$\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2$
e_7	$\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5$	$-\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$	$-\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2$
e_8	e_1	e_2	e_3	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$

pseudo case	e_5	e_6	e_7	e_8
e_1	$-\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5$	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	e_1
e_2	$\frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7$	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	e_2
e_3	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$	e_3
e_4	$\frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_8$	$\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$	$\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2$	$-\frac{1}{2}e_4 - \frac{\sqrt{3}}{2}e_5$
e_5	$\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$	$-\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$	$\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$	$\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$
e_6	$\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$	$-\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$	$-\frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_8$	$-\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$
e_7	$\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2$	$\frac{1}{2}e_3 - \frac{\sqrt{3}}{2}e_8$	$-\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$	$\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$
e_8	$-\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	$-\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7$	$-\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$-e_8$

(AII) Examples of triality group and derivation in the pseudo octonion algebra (the pseudo octonion algebra は Okubo algebra of eight dimensional matrix と呼ばれる) この代数は単位元がない. $(e_ie_j)e_i = e_j,\ < e_i, e_j >= \delta_{ij}$.

(AII):(pseudo octonion algebra), with respect to basis $e_1, \cdots e_8$.
(1)$(e_1, e_2, e_3), [e_2, -e_1, 0]$, (2)$(e_1, e_4, \frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7), p = [e_4, \frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2, 0]$, (3)$(e_1, e_5, -\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7), [e_5, \frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2, 0]$, (4)$(e_1, e_6, \frac{1}{2}e_5 + \frac{\sqrt{3}}{2}e_4), [e_6, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2, 0]$, (5)$(e_1, e_7, \frac{\sqrt{3}}{2}e_5 - \frac{1}{2}e_4), [e_7, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2, 0]$, (6)$(e_2, e_4 - \frac{\sqrt{3}}{2}e_7 + \frac{1}{2}e_6)$, $[e_4, \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2, 0]$, (7)$(e_2, e_5, \frac{1}{2}e_7 + \frac{\sqrt{3}}{2}e_6), [e_5, \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2, 0]$, (8)$(e_2, e_6, \frac{\sqrt{3}}{2}e_5 - \frac{1}{2}e_4), [e_6, -\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2, 0]$ (9)$(e_2, e_7, -\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5), [e_7, -\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2, 0]$,

(10)$(e_3, e_4, \frac{\sqrt{3}}{2}e_4+\frac{1}{2}e_5), [e_4, \frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8, 0]$, (11)$(e_3, e_5, \frac{\sqrt{3}}{2}e_5-\frac{1}{2}e_4), [e_5, \frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8, 0]$, (12)$(e_3, e_6, -\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7), [e_6, \frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8, 0]$, (13)$(e_3, e_7, -\frac{\sqrt{3}}{2}e_7+\frac{1}{2}e_6)$,
$[e_7, \frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8, 0]$, (14)$(e_4, e_4, -\frac{1}{2}e_8+\frac{\sqrt{3}}{2}e_3), [e_3, -\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8, 0]$
(15)$(e_4, e_5, \frac{\sqrt{3}}{2}e_8+\frac{1}{2}e_3), [e_5, -e_4, 0]$, (16)$(e_4, e_6, \frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2), [e_6, \frac{\sqrt{3}}{2}e_5+\frac{1}{2}e_4, 0]$,
(17)$(e_4, e_7, -\frac{\sqrt{3}}{2}e_2+\frac{1}{2}e_1), [e_7, \frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5, 0]$, (18)$(e_4, e_8, -\frac{\sqrt{3}}{2}e_5-\frac{1}{2}e_4)[e_5, -e_3, 0]$,
(19)$(e_5, e_5, \frac{\sqrt{3}}{2}e_3-\frac{1}{2}e_8), [e_3, \frac{\sqrt{3}}{2}e_8-\frac{1}{2}e_3, 0]$, (20)$(e_5, e_6, \frac{\sqrt{3}}{2}e_2-\frac{1}{2}e_1), [e_6, \frac{1}{2}e_5-\frac{\sqrt{3}}{2}e_4, 0]$, (21)$(e_5, e_7, \frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2), [e_7, -\frac{\sqrt{3}}{2}e_4+\frac{1}{2}e_5, 0]$,
(22)$(e_5, e_8, \frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5), [e_4, e_3, 0]$, (23)$(e_6, e_6, -\frac{1}{2}e_8-\frac{\sqrt{3}}{2}e_3), [e_3, -\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8, 0]$,
(24)$(e_6, e_7, \frac{\sqrt{3}}{2}e_8-\frac{1}{2}e_3), [e_7, -e_6, 0]$, (25)$(e_6, e_8, -\frac{\sqrt{3}}{2}e_7-\frac{1}{2}e_6), [e_7, e_3, 0]$,
(26)$(e_7, e_7, \frac{1}{2}e_8+\frac{\sqrt{3}}{2}e_3), [e_3, -\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8, 0]$, (27)$(e_7, e_8. -\frac{1}{2}e_7+\frac{\sqrt{3}}{2}e_6)$,
$[e_6, -e_3, 0]$, (28)$(-e_8, -e_8, -e_8), [e_1, -e_1, 0]$. where $e_8e_8 = -e_8$. (e_8 が単位元的な役割です). pseudo octonion algebra は単位元を持たない 8 次元の代数系でありその derivations が作る代数は A_2 型の 8 次元の単純リー代数です.

Note that if we define a product by $[\frac{1}{2}e_j, \frac{1}{2}e_k] = \sum_{l=1}^{8}\ f_{jkl}(\frac{1}{2}e_l)$, where the notation f_{jkl} refer ([G], [O], [K.3]), then it makes the Lie algebra of type A_2. 一方 para octonion algebra (the conjugation algebra of octonion) の derivation の場合は 14 次元の G_2type です.

Appendix II (Okubo algebra について)

pseudo octonion algebra $\mathbf{O_p}$ の別名である Okubo algebra について述べたいと考えます.

A を任意の traceless 3×3 matrix の集合とする. つまり

$A = \{X | X \in Mat\ (3,3;F),\ trace\ X = 0\}$, ただし $ch\ F \neq 2, 3$.

このとき, $\mu\ and\ \nu$ を次のような複素数とする, $\overline{\nu}$ は共役元.

$3\mu\ \nu = \mu + \nu = 1$, or $\mu = \overline{\nu} = \frac{1}{6}(3 \pm \sqrt{-3})$. そして $\forall X, Y \in A$ に対して

(♣) $XY = \mu X * Y + \nu Y * X - \frac{1}{3}(trace\ X * Y)E_3$

ただし右辺の $X * Y$ は行列の普通の積, E_3 は 3×3 単位行列 and $< X, Y >= \frac{1}{6}\ trace(X * Y)$ により XY なる new product と inner product $< X, Y >$ を定義する.

この (A, XY) は 8 次元代数であり, 単位元を持たない, 次の性質を有する;
$trace\ (XY) = 0,\ (XY)X = X(YX) =< X, X > Y,\ and$
$< XY, XY >=< X, X >< Y, Y >$ (合成代数), $< XY, Z >=< X, YZ >$.

この new product (♣) を持つ (A, XY) を Okubo algebra と呼びます.

そして, この (A, XY) で $[X, Y] = XY - YX$ とリー積を定義すると, 8 次元の A_2 type ($= sl(3)$) のリー代数となり, (A, XY) の derivation 全体はまた A_2 type のリー代数を生成します. 前述した $\mathbf{O}$ のような para octonion はこの性質の $[X, Y] = XY - YX$ が Malcev algebra でありリー代数となりません, しかし $Der\ \mathbf{O}$ は G_2 type のリー代数を生成します.

(♣) の積を持つ Okubo algebra は Lie admissible algebra, and flexible algebra です．Furthermore, real Okubo algebra is a division algebra, because $(\frac{X}{<X,X>}Y)X = Y$，もう少し一般化すると formally real symmetric composition algebra is a division algebra.

Okubo algebra は $ch\ F = 3$ の場合を含めて以下の様に一般化して構成可能です（筆者の考察です）．

A を traceless $n \times n$ matrix $(n \geq 3)$ とする．つまり

$A = \{X | X \in Mat(n,n;F),\ trace\ X = 0\}$, and $x^n = 1$ の primitive root を ω とするとき，

$$XY = \omega X * Y - \omega^{n-1} Y * X - (\frac{\omega - \omega^{n-1}}{n}) trace(X * Y) E_n$$

where 右辺の $X * Y$ は行列の普通の積です．勿論 $trace(XY) = 0$.

この場合 (A, XY) は $n^2 - 1$ 次元の単位元を持たない代数系です.

特に $[X,Y] = XY - YX$ でリー積を定義すると $(A,\ [X,Y])$ は A_{n-1}type $(= sl(n,F))$ のリー代数を生成します．従って (A, XY) は Lie admissible algebra

と呼ばれる代数系を構成します．また $< X, Y >= \frac{1}{2n}(trace\ X * Y)$ ただし $< E_n, X >= 0$ と内積を定義すると $< [X,Y], Z > + < Y, [X,Z] >= 0$ が成り立ちます．特に $n = 3$, and if we set

$$\mu = \frac{\omega}{\omega - \omega^2},\ \nu = \frac{-\omega^2}{\omega - \omega^2},$$

then we obtain the Okubo algebra (that is, the pseudo octonion algebra).

Mathematical physics のある特徴ずけに現れる Okubo algebra の大久保進氏 (1930-2015) は仁科賞・Wigner prize 受賞者であり，筆者は多数の共同研究（共著論文）を作成させていただきました.

図式化・簡略的に我々が最近考察しています事柄;

para octonion, pseudo octonion (or Okubo algebra)$-->$ symmetric composition algebra $-->$normal triality algebra,

Jordan, Lie, the conjugation algebra of structurable algebra$-->$ normal-triality algebra,

Under a certain condition in flexible, alternative algebra$-->$ normal triality algebra.

triple system or normal triality algebra$==>$construction of Lie (super)algebra.

物理+ 代数　$=$ mathematical algebra (数理代数学) $\cdots$ 将来の夢

Appendix III (歴史的な事柄–私の観点での話です).

Section 3 において述べた (a) $A = \mathbf{O} \otimes \mathbf{O}$, (b) $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ c) $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$ の場合，これらは例外型の単純リー代数の構成に関して現れる代数系です．これらの起源については誰に着目するかにより数学史的な観点が異なります．(a certain viewpoint without using concept of root sytems and Cartan matrix) ーー今後の課題ですーー

Jordan algebra が現れる幾何学的な事柄；

∗ E.Cartan or N.Jacobson による　$Der\mathbf{O}$ から　14 次元の G_2.

∗ C.Chevalley and R.D.Schafer による $Der(H_3(\mathbf{O}))$ から 52 次元の F_4.

∗ H.Freudenthal による $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ から $P \times Q$ and $\{P, Q\}$ concept のもとでの 56 次元の metasymmmplectic geometry に現れる三項系代数.

∗ Tits or Freudenthal による　$A = \mathbf{O} \otimes \mathbf{O}$, から magic square table.

∗ Tits による　182 次元の vector space $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$ から E_8.

∗ Tits による Cayley algebra **O** の微分の拡張から三対原理 (local triality).

∗ Springer による Jordan algebra から algebraic groups.

∗ Loos or Satake による　certain Jordan triple systems から　対称空間又は bounded symmmetric domains.

* Koeher, Kaup or Upmeier, T.Nomura による Jordan algebra から harmonic analysis ($\mathbf{C}^*$ algebra) and Toeplitz operators.

* Encyclopedia of Mathematics(2001,Kluwer Academic Publishers) の Lie triple system の項目にはリー三項系とその standard embedding Lie algebras の事柄が簡潔に説明されています. また Jordan and Lie algebras の関係する history については最近の筆者の ([K.4]) にも述べていますので興味のある方は参照してください.

以上思いつくままの個人的な見解 (意見・実例) の視点ですが 次のような対応が存在していると考えています. （nonassociative algebras world においての話です）

代数概念＜＝＝＝＝＞幾何学的概念

代数系をどのような立場で視るかによって焦点 (視座) が異なりますがここでは非結合的代数系の観点からの話です. 特にここで述べた三対原理は Jacobson, Freudenthal, Yokota 等の視点からではなく Tits の観点からの論及が発端であることに注意してください.

２０世紀に誕生した Lie algebras and Jordan algebra という概念の数学史を２１世紀に生きる我々がほんの少しだけ論究したいという欲求のためにーーー

もし許されるならば時々この様な研究ノート (数学の部分は英文で, 数学史評論は日本文, タイトルは両言語) の断片を残りの余白を借りて後日のために記録として残しておきたいと考えています.

References

[B]　W.Bertam, Complex and quaternionic structures on symmetric spaces-correspondence with Freudenthal-Kantor triple systems, Sophia Kokyuroku in Kathematics, vol. 45, Theory of Lie Groups and Manifolds, (2002) 57-76.

[G]　M.Gell-mann, Symmetry of Baryons and Mesons, Phys.Rev.125(1962), 1067-1084.

[K.1] N.Kamiya, On generalized structurable algebras and Lie related triple, Adv.Clifford Algebras,5(1995)127-140.

[K.2] N.Kamiya, A construction of Quandles associated with Quadratic algebras II, RIMS Kokyuroku(Kyoto Univ.) vol.2188,(2021) 34-44.

[K.3] N.Kamiya, On triality relations of certain eight dimensional algebras, RIMS, Kokyuroku (Kyoto Univ.) vol.2229, (2022) 52-63.

[K.4] N.Kamiya, On certain algebraic structures associated with Lie (super)algebras, Springer, Proc. Math. and Statistics, vol.427 (2023) 65-79.

[K.5] N.Kamiya, A structure theory of Freudenthal-Kantor triple systems, J. Algebra, vol.110, (1987) no.1, 108-123.

[K.6] N.Kamiya, 日本で出版されたリー代数という本の著作について, 津田塾大学数学・計算機科学研究所報 vol. 45, (2024) 148-159.

[K-O.1] N.Kamiya and S.Okubo, Triality of structurable and pre-structurable algebras, J. Alg. vol. 416, (2014)58-83.

[K-O.2] N.Kamiya and S.Okubo, Algebras satisfying triality and S_4 symmetry, Arxiv.1503.00614, A.G.G.vol.33,(2016)1-92

[K-O.3] N.Kamiya and S.Okubo, A triality group of nonassociative algebras with involution, Arxiv.1609.05892, A.G.G. vol 35,(2018) 113-168.

[K-O.4] N.Kamiya and S.Okubo, Symmetry of Lie algebras associated with (ε, δ) Freudenthal-Kantor triple systems. Proc.Edinb.Math.Soc. vol.89, (2016) 169-192.

[O] S.Okubo, *Introduction to Octonion and other nonassociative algebras in Physics,* Cambridge Univ. press.(1995).

[S] R.D.Schafer, *An introduction to nonassociative algebras,* Academic press, (1966).

[T] J.Tits, Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles, I. construction, Indag. Math. 28(1966),223-237.

[Tô] S.Tôgô, リー代数, 槙書店,(1984).

Current address;
Noriaki Kamiya
shigekamiya@outlook.jp
Japan, Chigasaki City chigasaki 1-2-47-201