

On certain algebraic structures associated with symmetric spaces

— 対称空間と三項系代数 —

Noriaki KAMIYA (神谷徳昭)

Center for Mathematical Sciences,
University of Aizu, 965-8580 , JAPAN (福島県立会津大学)

Abstract In this work we discuss a characterization and a construction of Lie (super)algebras from (ϵ, δ) -Freudenthal Kantor triple systems of certain types. Also we give examples of such triple systems. From these results, we construct symmetric (super)spaces associated with some Lie (super)algebras.

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Introduction

This paper is an announce of new idea and a survey note for nonassociative algebras and triple systems, in particular it will be explained a history about our research works mainly. And this work is in close contact with a symmetric space with complex structure. That is, in related with curvature and torsion tensors of differential geometry.

From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [11], [39], [63], [64], [65]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([5], [10]-[13], [15], [26], [28], [29], [38], [47], [48], [52], [55], [56], [57] [58], [60]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [6]-[8], [20], [23], [24], [33], [38], [43]-[46], [49], [51]) by using the standard embedding method ([22], [41], [42], [50], [54]). In particular, the generalized

Jordan triple system of second order, or $(-1, 1)$ -Freudenthal Kantor triple system (for short $(-1, 1)$ -FKTS), is a useful concept ([13]-[21], [34]-[37], [40], [53]) for the constructions of simple Lie algebras, while the $(-1, -1)$ -FKTS plays the same role ([6], [22], [25], [27], [58], [59]) for the construction of Lie superalgebras, while the δ -Jordan Lie triple systems act similarly for that of Jordan superalgebras ([23], [24], [49]). Specially, we have constructed a model of Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ ([25]).

As a special comment of this introduction, we provide well-known results due to O. Loos(1939-2020) as follows; if A is a unital commutative Jordan algebra, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system, i.e., it satisfies the two relations $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a fundamental identity), $\{xyz\} = \{zyx\}$ and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system.

Briefly summarizing this article, we will generalize these results and construct Lie (super)algebras associated with a generalized Jordan triple system. Toward to its applications, we will give a construction of symmetric spaces with an almost complex structure.

Roughly describing, we have an illustration;

Algebraic structures \iff Geometric structures.

For examples, symmetric, R-symmetric, and homogeneous spaces, totally geodesic manifold, symmetric domains, etc.

1 Preamble and Definitions

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3 , unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \rightarrow V$, $(x, y, z) \mapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \quad (1)$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \quad (2)$$

where $L(a, b)c := (abc)$ and $K(a, b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba), \text{ i.e., } K(a, c)b = 0. \quad (3)$$

The JTS is a special case in the GJTS of 2nd order since $K(x, y) \equiv 0$.

We next can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [18], [22], [28], [54] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \quad (4)$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0, \quad (5)$$

where

$$L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \quad (6)$$

is called an (ε, δ) -*Freudenthal - Kantor triple system* (for short (ε, δ) -FKTS). An (ε, δ) -FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized FKTS* (for short GFKTS), while the identity (5) is called the *second order condition*.

Remark From the relation Eq. (6), we note that

$$K(b, a) = -\delta K(a, b). \quad (7)$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha \langle x, y \rangle z + \beta \langle y, z \rangle x + \gamma \langle z, x \rangle y,$$

where $\langle x, y \rangle$ is a bilinear form such that $\langle x, y \rangle = \kappa \langle y, x \rangle$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$.

From now on we will mainly consider this type of triple system.

An (ε, δ) -FKTS is said to be *balanced* if there is a bilinear form $\langle x, y \rangle \in \Phi^*$ such that $K(x, y) = \langle x, y \rangle Id$, that is, $\dim \{K(x, y)\}_{span} = 1$ holds.

Remark We note that a balanced triple system (i.e., it fulfills $K(x, y) = \langle x, y \rangle Id$) is unitary, since $Id \in \{K(x, y)\}_{span}$.

Triple products are denoted by (xyz) , $\{xyz\}$, $[xyz]$ and $\langle xyz \rangle$ upon their suitability.

Remark We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$ -FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([6], [13]-[18], [22], [25], [27], [36],

[54]).

For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc]$, $a, b, c \in V$ is called a δ -Lie triple system (for short δ -LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \quad (8)$$

where $a, b, x, y, z \in V$. An 1-LTS is a LTS while a -1 -LTS is an *anti-LTS*, by ([14]). Note that the set $L(V, V)$ of all left multiplications $L(x, y)$ of V is a Lie subalgebra of $Der V$, where we denote by $L(x, y)z = [xyz]$, and $[X, Y] = XY - YX$, $\forall X, Y \in L(V, V)$.

Proposition 1.1 ([14], [15], [22]) *Let $(U(\varepsilon, \delta), \langle xyz \rangle)$ be an (ε, δ) -FKTS. If J is an endomorphism of $U(\varepsilon, \delta)$ such that $J \langle xyz \rangle = \langle JxJyJz \rangle$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon, \delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] := \langle xJyz \rangle - \delta \langle yJxz \rangle + \delta \langle xJzy \rangle - \langle yJzx \rangle. \quad (9)$$

Corollary ([13]) *Let $U(\varepsilon, \delta)$ be an (ε, δ) -FKTS. Then the vector space $T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} L(a, d) - \delta L(c, b) & \delta K(a, c) \\ -\varepsilon K(b, d) & \varepsilon(L(d, a) - \delta L(b, c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}. \quad (10)$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(U(\varepsilon, \delta)) = D(T(\varepsilon, \delta), T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta)$, associated with $T(\varepsilon, \delta)$ where $D(T(\varepsilon, \delta), T(\varepsilon, \delta))$ is the set of inner derivations of $T(\varepsilon, \delta)$;

$$D(T(\varepsilon, \delta), T(\varepsilon, \delta)) := \left\{ \begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix} \right\}_{span},$$

$$T(\varepsilon, \delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U(\varepsilon, \delta) \right\}_{span}.$$

We use the following notation:

$$\mathbf{k} := \{K(x, y) \in End U(\varepsilon, \delta) | x, y \in U(\varepsilon, \delta)\} \text{ and}$$

$$\{EFG\} := EFG + GFE, \quad \forall E, F, G \in \mathbf{k}.$$

Then, we may make the structure of a JTS \mathbf{k} with respect to the triple product $\{EFG\} \in \mathbf{k}$ ([20]). Also the JTS \mathbf{k} is called *nondegenerate* if $K(x, y) = 0$ for any $y \in U(\varepsilon, \delta)$ implies $x = 0$. Hence we have the following Proposition.

Proposition 1.2 ([15], [31]) *Let U be a unitary (ε, δ) -FKTS and $L(U)$ be the standard embedding Lie (super)algebra associated with U . Then the following are equivalent:*

- (i) U is simple,
- (ii) the Lie (super)algebra $L(U)$ is simple,
- (iii) the JTS $\mathbf{k} := \{K(a, b)\}_{span}$ is simple and nondegenerate.

Remark We note that $L(U) = L(U(\varepsilon, \delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra such that $U(\varepsilon, \delta) \oplus U(\varepsilon, \delta) = L_{-1} \oplus L_1 = T(\varepsilon, \delta)$ (δ -LTS), $L_{-2} = \mathbf{k}$ (JTS) and $D(T(\varepsilon, \delta), T(\varepsilon, \delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the derivation of $T(\varepsilon, \delta)$) equipped with $[L_i, L_j] \subseteq L_{i+j}$ and $L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In Introduction, we had used the notation $g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems without using root systems (for a Lie superalgebra, refer to ([12], [52])).

2 A generalized curvature and torsion tensors

Let $L = L(U(\varepsilon, \delta)) = L(W, W) \oplus W$ be the Lie (super)algebra defined from a δ LTS as in section one, that is, the δ -LTS $W = T(\varepsilon, \delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon, \delta)$ (as L_{-1} has the structure of a (ε, δ) -FKTS).

We introduce a generalization of covariant derivative ∇ in differential geometry as follows; $\nabla : L \rightarrow \text{End } L$ defined by

$$\begin{aligned} \nabla_X Y &= [X, Y] = -\delta[Y, X], \\ \nabla_X [Y, Z] &= [YZX] = -\delta[ZYX], \\ \nabla_{[X, Y]} Z &= -[XYZ] = -\delta[YXZ], \\ \nabla_{[X, Y]} [V, Z] &= [[V, Z][X, Y]] = -\delta[[X, Y][V, Z]], \end{aligned}$$

for any $X, Y, Z \in W$.

Furthermore, a generalized curvature tensor defined by

$$C_\delta(X, Y) = \nabla_X \nabla_Y - \delta \nabla_Y \nabla_X - \nabla_{[X, Y]} \quad (11)$$

is identically zero, i.e., $C_\delta(X, Y) = 0$ in L , for any $X, Y \in W$. Indeed, we demonstrate the proof below.

First we calculate

$$C_\delta(X, Y)Z = (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)Z - \nabla_{[X, Y]}Z$$

$$\begin{aligned}
&= \nabla_X[Y, Z] - \delta \nabla_Y[X, Z] + [XYZ] \\
&= [YZX] - \delta[XZY] + [XYZ] \\
&= [YZX] + [ZXY] + [XYZ] = 0.
\end{aligned}$$

Second, it follow

$$\begin{aligned}
C_\delta(X, Y)[V, Z] &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)[V, Z] - \nabla_{[X, Y]}[V, Z] \\
&= [X, [VZY]] - \delta[Y, [VZX]] + \delta[[X, Y], [V, Z]] \\
&= [X, L(V, Z)Y] - \delta[Y, L(V, Z)X] - L(V, Z)[X, Y] = 0
\end{aligned}$$

(by $[Y, L(V, Z)X] = -\delta[L(V, Z)X, Y]$ and $[[X, Y], [V, Z]] = -\delta[[V, Z], [X, Y]]$) for any $X, Y, Z, V \in T(\varepsilon, \delta)$.

However a generalized torsion tensor defined by

$$S_\delta(X, Y) = \nabla_X Y - \delta \nabla_Y X - [X, Y] \quad (12)$$

is not zero, since it gives $S_\delta(X, Y) = [X, Y] - \delta[Y, X] - [X, Y] = [X, Y]$.

To later section we next define the Nijenhuis operator

$$N(X, Y) = [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY],$$

where J is an almost complex structure on W , this concept (the case of $\delta = 1$) is appeared in [32].

3 Examples of (ε, δ) -JTS

We consider here examples of the particular case when $K(x, y) \equiv 0$ (identically), that is of an (ε, δ) -JTS, or of (α, β, γ) triple systems.

Example 3.1 Let V be a vector space with a symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$$

defines on V a $(-1, 1)$ -JTS.

Note that $(-1, 1)$ -JTS is same as the JTS.

Example 3.2 Let V be a vector space with an anti-symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$$

defines on V a $(1, -1)$ -JTS.

Example 3.3 Let V be a vector space with a symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z - \langle y, z \rangle x$$

defines on V a $(-1, -1)$ -JTS.

Example 3.4 Let V be a vector space with an anti-symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z - \langle y, z \rangle x$$

defines on V a $(1, 1)$ -JTS.

Example 3.5 Let V be a set of alternative matrix $Alt(n, \Phi) = \{x \mid x^t = -x\}$. Then

$$\langle xyz \rangle = x^t y z - \varepsilon z^t y x, \quad \text{where } \forall x, y, z \in V$$

defines on V a $(\varepsilon, -\varepsilon)$ JTS.

Proposition 3.1 *Let $(U, \langle xyz \rangle)$ be an (ε, δ) -JTS. Then the triple system is a δ -LTS with respect to the new product*

$$[xyz] = \langle xyz \rangle - \delta \langle yxz \rangle. \quad (13)$$

In the next section 5 subsection we study the case of an (ε, δ) -FKTS, but we give first two examples which are not (ε, δ) -JTS as it follows.

Proposition 3.2 *Let $(U, \langle xyz \rangle)$ be a triple system with $\langle xyz \rangle = \langle y, z \rangle x$ and $\langle x, y \rangle = -\varepsilon \langle y, x \rangle$. Then this triple system is an (ε, δ) -FKTS.*

Proposition 3.3 ([16], [18]) *Let U be a balanced $(1, 1)$ -FKTS satisfying $\langle \langle xxx \rangle, x \rangle \equiv 0$ (identically) and $\langle x, y \rangle$ is nondegenerate. Then U has a triple product defined by*

$$\langle xyz \rangle = \frac{1}{2}(\langle y, x \rangle z + \langle y, z \rangle x + \langle x, z \rangle y). \quad (14)$$

On the other hand, note that the balanced $(1, 1)$ -FKTS induced from an exceptional Jordan algebra is closely related to the 56 dimensional meta symplectic geometry due to H. Freudenthal ([13], [15], [16] and the earlier references therein). Also the correspondence of a quaternionic symmetric space and the balanced $(1, 1)$ FKTS has been studied in ([5]).

4 Symmetric spaces associated with (ε, δ) Jordan triple systems

This article (section) means that the symmetric space in sense of Prof. Wolfgang Bertram [Lecture Note in Math. "The geometry of Jordan and Lie structures" Springer, vol.1754 (2000)]. In his book, the following is described;

a) the category of germs of symmetric space with invariant almost complex structure is equivalent to the category of Lie triple systems with invariant complex structure,

b) the category of germs of symmetric space with invariant polarizations is equivalent to the category of Lie triple system with invariant polarization.

Hence from these results, it seems that it is important to construct a Lie triple system from a Jordan triple system (abbreviated JTS). Therefore we construct δ -Lie triple systems associated with a $(\varepsilon, -\varepsilon)$ -JTS of more a general case (when $\varepsilon\delta = -1$).

From now on, let V be a (ε, δ) -JTS with $\varepsilon = -\delta$, that is, $\varepsilon\delta = -1$ and \widehat{W} be a subset in the δ -LTS $W = T(\varepsilon, \delta)$ satisfying

$$\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-, \text{ where } \widehat{W}_+ := \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in V \right\}, \widehat{W}_- := \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \middle| x \in V \right\}.$$

Then we set $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $i = \sqrt{-1}$, and we get $JW_+ = W_-$, $JW_- = W_+$.

From a special case of section one, for δ -LTS W with $K(x, y) \equiv 0$, we have

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] = \begin{pmatrix} (L(x_1, y_2) - \delta L(y_1, x_2))z_1 \\ (\varepsilon L(y_2, x_1) - \varepsilon\delta L(x_2, y_1))z_2 \end{pmatrix}.$$

Thus for \widehat{W}_+ and \widehat{W}_- respectively, we obtain

$$\left[\begin{pmatrix} x \\ x \end{pmatrix} \begin{pmatrix} y \\ y \end{pmatrix} \begin{pmatrix} z \\ z \end{pmatrix} \right] = \begin{pmatrix} L(x, y)z - \delta L(y, x)z \\ \varepsilon L(y, x)z - \varepsilon\delta L(x, y)z \end{pmatrix} \in \widehat{W}_+,$$

$$\left[\begin{pmatrix} x \\ -x \end{pmatrix} \begin{pmatrix} y \\ -y \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} \right] = \begin{pmatrix} -L(x, y)z + \delta L(y, x)z \\ \varepsilon L(y, x)z - \varepsilon\delta L(x, y)z \end{pmatrix} \in \widehat{W}_-.$$

Also, we have $[\widehat{W}_- \widehat{W}_- \widehat{W}_+] \subset \widehat{W}_+$ and $[\widehat{W}_+ \widehat{W}_+ \widehat{W}_-] \subset \widehat{W}_-$ etc.

Here we define for $X, Y, Z \in \widehat{W}_+$ or $X, Y, Z \in \widehat{W}_-$,

$$R(X, Y)Z = -[X, Y, Z] = -[[X, Y], Z],$$

$$T(X, Y)Z = T(X, Y, Z) = -\frac{1}{2}(R(X, Y)Z - JR(X, J^{-1}Y)Z).$$

Hence, for example $T(X, Y)Z = \begin{pmatrix} -L(x, y)z \\ L(x, y)z \end{pmatrix}$ for any $X, Y, Z \in \widehat{W}_-$.

Theorem 4.1. *Under the above assumption and $\sqrt{-1} \in \Phi$ (base field), then we have,*

i) \widehat{W}_+ and \widehat{W}_- are a δ -LTS with respect the product $[XYZ]$,

- ii) \widehat{W}_+ and \widehat{W}_- are a $(\varepsilon, -\varepsilon)$ -JTS with respect to the product $T(X, Y, Z)$,
- iii) \widehat{W} is twisted ($[JXYZ] = -[XJYZ]$) and almost complex with J ,
- iv) $N(X, Y)$ is vanished, where $N(X, Y) := [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY]$, (Nijenhuis operator),

Note that $JT(X, Y, Z) = T(JX, Y, Z) = -T(X, JY, Z)$ hold for $X, Y, Z \in \widehat{W}_\pm$, i.e., this is hermite in the sense of W. Bertram.

Remark Note that there is no the addition on $\widehat{W}_+ + \widehat{W}_-$, but there is the multiplication $[XYZ]$ on \widehat{W} and the addition on \widehat{W}_+ or \widehat{W}_- .

Similar if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Then we have \widehat{W} is twisted and polarized, i.e., $J^2 = Id$ on \widehat{W} .

If $J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, then it is straight with $T(X, Y) = 0$,

if $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then it is straight with $T(X, Y) = 0$ i.e., $[JXYZ] = [XJYZ]$.

Thus we note that "there are the concept of invariant ($J[XYZ] = [XYJZ]$), automorphism and derivation on \widehat{W}_\pm ".

If $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \implies N(X, Y) = 0$ (identically zero),

if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \implies N(X, Y) \neq 0$,

if $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \implies$ auto, but $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not auto,

if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \implies$ auto, but $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not auto.

Sumarizing these results, for the structure J of \widehat{W} we have the table as follows, ($\varepsilon = -1, \delta = 1$)

	almost complex	polarized
twisted auto	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ (type I)	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
twisted anti-auto	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (type II)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
straight	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ $T(X, Y) \equiv 0$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $T(X, Y) \equiv 0$

where the anti-automorphism is denoted by $[JXJYJZ] = -[XYZ]$ on \widehat{W} .

Note that twisted and invariant \implies derivation.

5 Examples of Lie (super)algebras associated with (ε, δ) Freudenthal-Kantor triple systems and B_3 type Lie algebra

5.1 Examples of simple Lie superalgebra

Example a) $C(n+1)$ type is of dimension $\dim C(n+1) = 2n^2 + 5n + 1$.

Let U be the set of matrices $M(1, 2n; \Phi)$. Then, by Example 3.2, it follows that the triple product

$$L(x, y)z = \langle xyz \rangle := \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$$

such that the bilinear form fulfills $\langle x, y \rangle = -\langle y, x \rangle$, is a $(1, -1)$ -JTS, since $K(x, y) \equiv 0$ (identically). Furthermore, the standard embedding Lie superalgebra is 3-graded and of $C(n+1)$ type. For the extended Dynkin diagram, we obtain

$$\begin{aligned} L_{-1} \oplus L_0 \oplus L_1 &:= \left\{ \left(\begin{array}{cc} L(a, b) & 0 \\ 0 & \varepsilon L(b, a) \end{array} \right) \middle| \varepsilon = 1 = -\delta \right\}_{span} \oplus \left\{ \begin{pmatrix} e \\ f \end{pmatrix} \right\}_{span} \cong \\ &\quad \otimes \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_n \quad \alpha_{n+1} \\ &\quad \parallel \quad > \quad \circ - \circ - - - - - \circ < = \circ \\ &\quad \otimes \alpha_0 \\ &= C(n+1) \text{ type } (\alpha_1 \otimes \text{deleted}). \end{aligned}$$

Also, we obtain

$$\begin{aligned} L_0 &:= \left\{ \left(\begin{array}{cc} L(a, b) & 0 \\ 0 & \varepsilon L(b, a) \end{array} \right) \middle| \varepsilon = 1 = -\delta \right\}_{span} \cong \\ &\quad \alpha_2 \quad \alpha_3 \quad \alpha_n \quad \alpha_{n+1} \\ &\quad \circ - \circ - - - - - \circ < = \circ \\ &= C_n \oplus \Phi Id (\alpha_1 \otimes \text{and } \alpha_0 \otimes \text{deleted}). \end{aligned}$$

Thus the last diagram is obtained from the extended Dynkin diagram of $C(n+1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \otimes$.

Example b) $B(n, 1)$ and $D(n, 1)$ type are of dimension $\dim B(n, 1) = 2n^2 + 5n + 5$ and $\dim D(n, 1) = 2n^2 + 3n + 3$, respectively.

Let U be the set of matrices $M(1, l; \Phi)$. Then, by straightfoward calculations, it follows that the triple product

$$L(x, y)z = \langle xyz \rangle := \frac{1}{2}(\langle x, y \rangle z - \langle y, z \rangle x + \langle z, x \rangle y)$$

such that the bilinear form fulfills $\langle x, y \rangle = \langle y, x \rangle$ is a $(-1, -1)$ -FKTS. Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(n, 1)$ type if $l = 2n + 1$, or of $D(n, 1)$ type if $l = 2n$. For the extended Dynkin diagram, we obtain from the results of § 1 the following.

For the case of $B(n, 1)$ type we have

$$\begin{aligned}
L_{-2} \oplus L_0 \oplus L_2 &:= D(T(-1, -1), T(-1, -1)) = \\
&\left\{ \left(\begin{array}{cc|cc} L(a, b) & \delta K(c, d) & & \\ -\varepsilon K(e, f) & \varepsilon L(b, a) & & \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong \\
&\quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \quad \quad \alpha_n \quad \alpha_{n+1} \\
&\quad \circ = > \otimes - \circ - - - - - \circ = > \circ \\
&= A_1 \oplus B_n \text{ type } (\alpha_1 \otimes \text{ deleted}).
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
L_0 &:= \left\{ \left(\begin{array}{cc|cc} L(a, b) & 0 & & \\ 0 & \varepsilon L(b, a) & & \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong \\
&\quad \alpha_2 \quad \alpha_3 \quad \quad \quad \alpha_n \quad \alpha_{n+1} \\
&\quad \circ - \circ - - - - - \circ = > \circ \\
&= B_n \oplus \Phi Id (\alpha_1 \otimes \text{ and } \alpha_0 \circ \text{ deleted}).
\end{aligned}$$

Thus the last diagram is obtained from the extended Dynkin diagram of $B(n, 1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \circ$.

Similarly, for the case of $D(n, 1)$ type we have $L_{-2} \oplus L_0 \oplus L_2 \cong A_1 \oplus D_n$, $L_0 \cong D_n \oplus \Phi Id$. We note that this triple system is balanced and with a complex structure of type II since $K(x, y) = \langle x, y \rangle Id = L(x, y) + L(y, x)$ (c.f. [32]).

In final of this section we note that the case of balanced is discussed in ([18], [28]). On the other hand, for the construction of simple exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 , refer to ([16], [18], [21]). Also, for the construction of simple Lie superalgebras $G(3), F(4), D(2, 1, \alpha), P(n), Q(n), H(n), S(n)$ and $W(n)$, refer to ([22], [25], [27]). Of course, these construction are created from the concept of triple systems without using systems of roots. Thus we can construct symmetric superspaces from anti-Lie triple systems as well to the cases of symmetric spaces.

Remark Summarizing, we have the imaging methods;

$$bilinear \ forms \ \rightarrow \ triple \ products \ \rightarrow \ symmetric \ (super)spaces$$

5.2 Constructions of B_3 -type Lie algebra

In this subsection, we will consider the constructions of simple B_3 -type Lie algebra associated with several triple systems (the case of $\varepsilon = -1$ and $\delta = 1$), more easily. That is, we will give several examples; (c) the case of a JTS (i.e., $(-1, 1)$ -FKTS with $K(x, y) \equiv 0$), (d) the case of a GJTS of 2nd order (i.e., $(-1, 1)$ -FKTS with $\dim\{K(x, y)\}_{span} = 1$), (e) the case of a GJTS of 2nd order (i.e., $(-1, 1)$ -FKTS with $\dim\{K(x, y)\}_{span} = 3$), (f) the case of a derivation induced from a JTS (i.e., a subalgebra of B_4 -type).

Example c) We study the case of $g_{-1} = U = Mat(1, 5; \Phi)$. Hereafter in this subsection, as a reason of traditional notation, we often would like to denote by g_i instead of L_i , ($i = 0, \pm 1, \pm 2$) and by $\{xyz\}$ instead of $\langle xyz \rangle$.

In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x^t y z + y^t z x - z^t x y, \quad \forall x, y, z \in g_{-1}$$

where ${}^t x$ denotes the transpose matrix of x .

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_3 -type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and a LTS $T(U) = g_{-1} \oplus g_1$. Thus, we have

$$g_0 = Der U \oplus Anti - Der U \cong B_2 \oplus \Phi H, \quad \text{where } H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Here in view of the relations $[S(x, y), L(a, b)] = L(S(x, y)a, b) + L(a, S(x, y)b)$, and $[A(x, y), L(a, b)] = L(A(x, y)a, b) - L(a, A(x, y)b)$ for all $L(a, b) \in End U$, when $\varepsilon = -1, \delta = 1$, we use the following notations;

$$Der U := \{L(x, y) - L(y, x)\}_{span},$$

$$Anti - Der U := \{L(x, y) + L(y, x)\}_{span},$$

$$g_0 = \left\{ \begin{pmatrix} L(x, y) & 0 \\ 0 & -L(y, x) \end{pmatrix} \right\}_{span} = \{S(x, y) + A(x, y)\}_{span}$$

where $S(x, y) := L(x, y) - L(y, x) \in Der U$, $A(x, y) := L(x, y) + L(y, x) \in Anti - Der U$.

Example d) Second, we study the case of $g_{-1} = U = Mat(2, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$ -FKTS) with $\dim\{K(x, y)\}_{span} = 1$ with respect to the product

$$\{xyz\} = x^t y z + z^t y x - z^t x y, \quad \forall x, y, z \in g_{-1}.$$

By straightforward calculations, it can be shown that the standard embedding Lie algebra $L(U) = g$ is a 5-graded B_3 -type Lie algebra with $g =$

$g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $\dim g_{-2} = \dim g_2 = \dim \{K(x, y)\}_{span} = 1$. Thus, we have

$$g_0 = Der U \oplus Anti - Der U \cong A_1 \oplus A_1 \oplus \Phi H, \text{ where } H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Furthermore, we obtain a LTS $T(U)$ of $\dim T(U) = \dim (g_{-1} \oplus g_1) = 12$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1 \cong Der T(U).$$

Also, in this case, we note that $T(U) = L(U)/Der T(U) = g/(g_{-2} \oplus g_0 \oplus g_2) (= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since $T(U)$ is a Lie triple system associated with g_{-1} .

Example e) Third, we study the case of $g_{-1} = U = Mat(1, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$ -FKTS) with respect to the product

$$\{xyz\} = x^t yz + z^t yx - y^t xz, K(x, y)z = \{xzy\} - \{yzx\}, \forall x, y, z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 5-graded B_3 -type Lie algebra with $g = g_{-2} \oplus \cdots \oplus g_2$ and $\dim g_{-2} = \dim g_2 = 3$. Thus, we have

$$g_0 = Der U \oplus Anti - Der U \cong A_2 \oplus \Phi H, g_{-2} = \{K(x, y)\}_{span} = Alt(3, 3; \Phi).$$

Furthermore, we obtain a LTS $T(U)$ of $\dim T(U) = \dim (g_{-1} \oplus g_1) = 6$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_3 \cong Der T(U).$$

This case $g_{-2} = \{K(x, y)\}_{span} = \mathbf{k}$ has the structure of a JTS (cf. section 2).

Remark Note that the cases (a) and (b) (resp. (e)) are $\delta = -1$ (resp. $\delta = 1$).

Remark For the root system $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$ and the highest root $-\rho = \{\alpha_1 + 2\alpha_2 + 2\alpha_3\}$ of the simple Lie algebra B_3 , we note that the case of (c) means $g_{-1} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$, $g_2 = 0$, and the case of (d) means $g_{-1} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3\}$, $g_{-2} = \{\alpha_1 + 2\alpha_2 + 2\alpha_3\}$, and the case of (e) means $\{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$, $g_{-2} = \{\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$.

Example f) Finally, we study the case of $g_{-1} = U = Mat(1, 7; \Phi)$. In this case, g_{-1} is a JTS (i.e., $(-1, 1)$ -FKTS with $K(x, y) \equiv 0$) with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy, \forall x, y, z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_4 -type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and $\dim g_{-2} = \dim g_2 = 0$. Thus, we have

$$\begin{aligned} g_0 &= \text{Der } U \oplus \text{Anti} - \text{Der } U \cong B_3 \oplus \Phi H, \\ \text{Der } U &= \{L(x, y) - L(y, x)\}_{\text{span}} = \text{Alt}(7, 7; \Phi) \cong B_3 \\ \text{Anti} - \text{Der } U &= \{L(x, y) + L(y, x)\}_{\text{span}} \cong \Phi H. \end{aligned}$$

This case is obtained from $\text{Der } U$ such that $U = \text{Mat}(1, 7; \Phi)$ with the JTS structure and so this derivation $\text{Der } U$ is a subalgebra of the B_4 -type Lie algebra associated with $g_{-1} = U = \text{Mat}(1, 7; \Phi)$.

6 A generalization case of (ε, δ) JTS

Let V be a (ε, δ) -FKTS and $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $i = \sqrt{-1}$. This concept is a generalization of (ε, δ) -JTS in section 4. Then we set $\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-$ satisfying

$$\widehat{W}_+ = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in V \right\}, \widehat{W}_- = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \middle| x \in V \right\}. \text{ Hence } \widehat{W} \subset W.$$

Theorem 6.1 *For the above \widehat{W} and $\varepsilon\delta = -1$, $\sqrt{-1} \in \Phi$, we have*

- i) \widehat{W}_\pm is a δ -LTS with respect the product $[XYZ]$ for any $X, Y, Z \in \widehat{W}_\pm$,
- ii) $T(X, Y, Z) = -\frac{1}{2}(R(X, Y)Z - JR(X, J^{-1}Y)Z)$ is a (ε, δ) -FKTS, where $R(X, Y)Z = -[XYZ]$, i.e., $(V, L(x, y)z) \leftrightarrow (\widehat{W}_\pm, T(X, Y, Z))$,
- iv) $N(X, Y)$ is vanished.

Indeed from the relation (10) in section one, for example, for \widehat{W}_- ,

$$\left[\begin{pmatrix} x \\ -x \end{pmatrix} \begin{pmatrix} y \\ -y \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} \right] = \begin{pmatrix} -L(x, y)z + \delta L(y, x)z - \delta K(x, y)z \\ \varepsilon L(y, x)z - \varepsilon\delta L(x, y)z - \varepsilon K(x, y)z \end{pmatrix},$$

$[XYZ] \in \widehat{W}_-$ and $JT(X, Y, Z) \in \widehat{W}_+$ for any $X, Y, Z \in \widehat{W}_-$, we obtain the results.

Remark. This generalized concept means that there is a symmetric (super)space associated with the (ε, δ) -FKTS, as same methods in section 4. However the details (type I and II) will be discussed in forthcoming paper.

7 Concluding Remarks

In this section, we give several references of mathematical physics in our works.

We note that there are applications toward the Yang-Baxter equations associated with triple systems ([26], [50]) and also toward the field theory associated with Hermitian triple systems ([61], [62]).

8 Appendix (history from a certain personal viewpoint)

For a Jordan river, we describe as follows:

This brief history (with respect to nonassociative algebras) is a story from author's personal aspect (judgement). It seems that triple systems (ternary algebras) have first been appeared from Prof. N. Jacobson and continued by Profs. O. Loos, K. Meyberg and E. Neher of students of Prof. M. Koecher in Germany, also have certain triple systems associated with the geometry of 56 dimensional due to Prof. Freudenthal have been studied by Prof. J. Faulkner (resp. K. Meyberg) of the student of Prof. N. Jacobson in U.S.A (resp. Germany).

On the other hand, there is a history;

H. Freudenthal \dashrightarrow K. Yamaguti or I. L. Kantor \dashrightarrow Author (N. Kamiya) or S. Okubo \dashrightarrow D. Mondoc (but these arrows are no students), however, Dr. Mondoc is only a student of Prof. Kantor.

Profs. O. Loos and E. Neher on the student of Prof. M. Koecher in Germany are working in Jordan triple systems and Jordan pairs. Profs. Kantor, S. Okubo and author(N. Kamiya) are studying in their generalizations, for example, refer to N. Kamiya and S. Okubo "Representation of (α, β, γ) triple systems," Linear and Multilinear Algebras, **58** no.5-6 (2010) 617-643. This history is a story without concept of root systems and Cartan matrix in Lie algebras, in particular, is a study for triple systems.

Note that there are a lot of mathematician in nonassociative algebras related with Lie algebras, but a little groups in triple systems or Jordan algebras. For example, Profs. E. Zelmanov, K. McCrimmon, B. Allison, V. Kac, I. Shestakov, H. Petersson, M. Racine, H. Asano, I. Satake, A. Elduque, C. Martinez, S. Okubo and author, may be, only a few. Furthermore in addition, the book "A Taste of Jordan Algebras" (Springer, 2003) written by Prof. K. McCrimmon of a student in N. Jacobson is described about a history of the Jordan river. It here emphasize that this historical survey of certain Jordan algebras at the end of the 20th century and the beginning of 21th century is my (author) aspect (viewpoint).

In final comments, for a generalization of numbers,

$$\mathbf{R} \rightarrow \mathbf{C} \rightarrow \mathbf{H} \rightarrow \mathbf{O}(\text{octonion}) \rightarrow \mathbf{H}_3(\mathbf{O})(\text{Jordan algebra of 27 dim}) \rightarrow$$

$$\begin{aligned} & \mathbf{M}(\mathbf{H}_3(\mathbf{O}))(\textit{metasymplectic geometry of 56 dim}) \rightarrow \\ & \mathbf{T}(\mathbf{H}_3(\mathbf{O}))(\textit{symmetric space of 112 dim}) \rightarrow \\ & \mathbf{E}_8(\textit{exceptional Lie algebra of 248 dim}), \end{aligned}$$

it seems that there are several group's researchers tradition with respect to these fields.

For algebraic structures of nonassociative subject (AMS classification 17) related with geometry, we may describe as follows, for example (in my opinion),

Jordan algebras researchers (E. Artin origin),

Lie algebras researchers (N. Jacobson origin).

In summarizing, we have the following diagrams (a generalization of complex and quaternionic numbers):

octonion, pseudo octonion algebras and triple systems \implies

Jordan algebras +Lie (super)algebras +symmetric composition algebras

\implies **mathematical algebras**

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These references are mainly papers for our study fields (as a survey article in these fields of nonassociative algebras, so we have a lot of references).

Current address

Noriaki Kamiya

Chigasaki city Chigasaki 1-2-47-201, Japan
e-mail; shigekamiya@outlook.jp