Theorema Egregium according to Gauss and Riemann

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§1. Historical introduction

In 1977, when it was 150 years celebration of Gauss's paper[G], on which symposium took place in Europe (, see [D]). In Japan, explicit form of theorema egregium due to Gauss was noted by Syoshichi Kobayashi (1932-2012), and Ichirou Satake (1927-2014), respectively (, see $[K]_1, [K]_2, [S]$).

In fact, Gauss was fifty years old in 1827, he published [G] in Latin, whose main results as follows.

(i) Theorema egregium

(ii) Theorema elegantissimun (i.e. the original form of Gauss-Bonnet theoreme)

(iii) A generalization of Legendre formula in spherical trigonometry

After Riemann's " Habilitationsvorlesumg " [R] in 1854, the following four Italian mathematicians contributed greatly to the development of modern differential geometry ;

Elwin Bruno Christoffel (1829-1900), Curbastro Gregorio Ricci (1853-1925), Luigi Bianchi (1856-1928), Tullio Levi-Civita (1873-1941).

At the present days, " Theorema egregium " can be proved by using ideas either of Riemann or of Cartan, as yet many books on differential geometry do not treat seriously the original proof due to Gauss himself. Though the explicit calculation by Gauss is extraordinary long, it seems to deserve to review. He used neither matrices nor determinants (i.e. no linear algebra), but his foremost sense for algebraic symmetry can be seen fully from his each formula.

§2. Statements of Gauss's theorma egregium and that of Riemann's

[GTE] Let $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$
be the first fundamental form of a surface in a three dimensional Euclidean space with respect to parameters (p,q), and let k be its Gaussian curvature. Then k can been expressed only by the first fundamental quantities E , F , G , $E_p (= \partial E/\partial p), E_q, F_p, F_q, G_p, G_q, E_{qq}, F_{qp}, G_{pp}$ *as follows.*

$$4(EG-F^2)^2k = -2(EG-F^2)(E_{qq}-2F_{qp}+G_{pp})+E(G_p^2-2F_pG_q+E_qG_q)$$
$$F(E_pG_q - E_qG_p - 2E_qF_q + 4F_pF_q - 2F_pG_p) + G(E_pG_p - 2E_pE_q + E_q^2).$$

... Formula itaque art. praec. sponte perducit ad egregium
Theorema. Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvatur in singulis punctis invariata manet.

((... Thus the formula of the preceding article leads to itself to the remarkable
Theorem. If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.))

[RTE] Let S = (S , g|S) be a submanifold of a Riemannian manifold M = (M , g). Let R be the curvature of M, and R_S *be the curvature of S.*

Let $II(X, Y) = \nabla_X Y - \nabla^0_X Y$

be the second fundamental form, where ∇ *and* ∇^0

denote the Levi-Civita connection of M and S, respectively.

Then we have $\langle (R - R_S)(X,Y)Z, W \rangle$

$= \langle II(X,Z), II(Y,W) \rangle - \langle II(Y,Z), II(X,W) \rangle$.

§3. Proof of [RTE]

Let TM $\longrightarrow M$, $TS \longrightarrow S$ *be tangent bundles, and let* $\nu(S) \longrightarrow S$ *be normal bundle of* S.

Since $\nabla_Y X - \nabla_X Y = [Y,X] = \overset{\circ}{\nabla}_Y X - \overset{\circ}{\nabla}_X Y$. *one sees that*

$II(Y,X) = \nabla_X Y + [Y,X] - (\overset{\circ}{\nabla}_X Y + [Y,X]) = II(X,Y)$ (*i.e. II is a symmetric form*).

It follows from $X\langle Y,Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle =$

$\langle \overset{\circ}{\nabla}_X Y, Z \rangle + \langle Y, \overset{\circ}{\nabla}_X Z \rangle$ *that*

$\langle Z, II(X,Y) \rangle = \langle II(X,Y), Z \rangle = -\langle Y, II(X,Z) \rangle = -\langle Y, II(Z,X) \rangle$

By similar way, $\langle X, II(Y,Z) \rangle = -\langle Z, II(X,Y) \rangle$ and $\langle Y, II(Z,X) \rangle = -\langle X, II(Y,Z) \rangle$.

Hence we have $\langle Z, II(X,Y) \rangle = \langle X, II(Y,Z) \rangle = -\langle Z, II(X,Y) \rangle$. Therefore $\langle Z, II(X,Y) \rangle = 0$ for arbitrary Z in $C^\infty(S;TS)$ (*i.e. $II(X,Y)$ is normal to S at point in M*).

Let ξ *be a section* $\in C^\infty(S;\nu(S))$, *which is normal to* S, *then there uniquely exists the Weingarten linear map (i.e. symmetric shape linear operator)*

$A_\xi : T_pS \longrightarrow T_pS$ *such that* $< A_\xi X, Y > = < \xi, II(X,Y) >$.

Furthermore, there uniquely exists a connection

(on $\nu(S))\overset{1}{\nabla}: C^\infty(S;TS)\times C^\infty(S;\nu(S)) \longrightarrow C^\infty(S;\nu(S))$

such that $\nabla_X \xi + A_\xi X = \overset{1}{\nabla}_X \xi$(*because, if* Y *is tangent to* S, *then* $< \nabla_X\xi, Y > + < A_\xi X, Y > = X < \xi, Y > - < \xi, \nabla_X Y > + < \xi, II(X,Y) > = - < \xi, \overset{0}{\nabla}_X Y > - < \xi, II(X,Y) > + < \xi, II(X,Y) > = 0$).

By the way, $\nabla_X(\nabla_Y Z) = \overset{0}{\nabla}_X(\nabla_Y Z) + II(X, \nabla_Y Z) = \overset{0}{\nabla}_X(\overset{0}{\nabla}_Y Z + II(Y,Z)) + II(X,\nabla_Y Z) = \overset{0}{\nabla}_X(\overset{0}{\nabla}_Y Z) + \overset{0}{\nabla}_X(II(Y,Z)) + II(X,\overset{0}{\nabla}_Y Z) + II(X, II(Y,Z)) = \overset{0}{\nabla}_X\overset{0}{\nabla}_Y Z + \nabla_X(II(Y,Z)) - II(X,II(Y,Z)) + II(X,\overset{0}{\nabla}_Y Z) + II(X,II(Y,Z)) = \overset{0}{\nabla}_X\overset{0}{\nabla}_Y Z + \overset{1}{\nabla}_X II(Y,Z) - A_{II(Y,Z)}X + II(X,\overset{0}{\nabla}_Y Z)$.

It follows from $II([X,Y],Z) = \nabla_{[X,Y]}Z - \overset{0}{\nabla}_{[X,Y]}Z$ *that* $(R - R_S)(X,Y)Z = R(X,Y)Z - R_S(X,Y)Z$

$= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z - (\overset{0}{\nabla}_X\overset{0}{\nabla}_Y Z - \overset{0}{\nabla}_Y\overset{0}{\nabla}_X Z - \overset{0}{\nabla}_{[X,Y]}Z)$

$= \overset{1}{\nabla}_X II(Y,Z) - A_{II(Y,Z)}X + II(X,\overset{0}{\nabla}_Y Z) - \overset{1}{\nabla}_Y II(X,Z) + A_{II(X,Z)}Y - II(Y,\overset{0}{\nabla}_X Z) - \nabla_{[X,Y]}Z + \overset{0}{\nabla}_{[X,Y]}Z$

$= (\,II(X,\overset{0}{\nabla}_Y Z) - II(Y,\overset{0}{\nabla}_X Z) - II([X,Y],Z) + \overset{1}{\nabla}_X II(Y,Z) - \overset{1}{\nabla}_Y II(X,Z))_{(\textit{normal to } S)}$

$- (\,A_{II(Y,Z)}X - A_{II(X,Z)}Y)_{(\textit{tangent to } S)}$.

This formula is called the Gauss-Codazzi equation.

For arbitrary vector fields (i.e. sections) X , Y , Z , W in $C^\infty(S;TS)$, *we obtain that*

$< (R - R_S)(X,Y)Z, W > = < -A_{II(Y,Z)}X, W > + < A_{II(X,Z)}Y, W >$
$= - < II(Y,Z), II(X,W) > + < II(X,Z), II(Y,W) >$.
This completes our proof of [*RTE*] .*Q.E.D.*

§4. Implication from [RTE] to [GTE]

Recall that the first fundamental form I(X,Y) =⟨X, Y⟩(, where X, Y in $C^\infty(M;TM)$) *is actually inner products, which are patched by the metric* g *of a Riemannian manifold* (M,g).

In the case of [GTE], since dim M = 3>dim S = 2 , dim $\nu(S)$ = 1, *we may write local coordinates* (x_1, x_2) (= (p,q) ; *parameters in* [*GTE*]),

and symbolically, we often write by $g_{11} = E, g_{12} = g_{21} = F, g_{22} = G$.

For the sake of brevity, write by $\partial_j = \partial/\partial x_j (j = 1,2)$. *For arbitrary* $X = a_{11}\partial_1 + a_{21}\partial_2$, $Y = a_{21}\partial_1 + a_{22}\partial_2$ *in* C^∞ $(S;TS)$, *since* $< \partial_i, \partial_j > = \delta_{ij}$, *and* $dim\xi = 1$, $< \xi, \xi > = 1$ (*for some* ξ *in* $C^\infty(S;\nu(S))$) *we have* $II(X,Y) = < \xi, II(X,Y) > \xi$, *and* $< X,X >< Y,Y > - < X,Y >^2 = (a_{11}a_{22} - a_{12}a_{21})^2$.

Now write by k_{ij} *the components of* A_ξ , *then the Gaussian curvature* k *of* S *is equal to* $det A_\xi = k_{11}k_{22} - k_{12}k_{21}$. *Since* $< A_\xi X, X > = a_{11}^2 k_{11} + a_{11}a_{21}(k_{21} + k_{12}) + a_{21}^2 k_{22}$,
$< A_\xi X, Y > = a_{12}(a_{11}k_{11} + a_{21}k_{21}) + a_{22}(a_{11}k_{12} + a_{21}k_{22})$,
$< A_\xi Y, Y > = a_{12}^2 k_{11} + a_{12}a_{22}(k_{21} + k_{12}) + a_{22}^2 k_{22}$, *we have*
$< A_\xi X, X >< A_\xi Y, Y > - < A_\xi X, Y >^2$
$= (a_{11}a_{22} - a_{12}a_{21})\big(a_{11}a_{12}k_{11}k_{21} + (a_{11}a_{22} - a_{21}a_{12})k_{11}k_{22} - a_{11}a_{12}k_{11}k_{12} + a_{21}a_{12}k_{21}^2 - a_{11}a_{22}k_{12}^2 + a_{21}a_{22}k_{21}k_{22} - a_{21}a_{22}k_{12}k_{22}\big)$.

Since A_ξ *is a symmetric linear operator* (*i.e.* $k_{12} = k_{21}$), *we have*
$< A_\xi X, X >< A_\xi Y, Y > - < A_\xi X, Y >^2$
$= (a_{11}a_{22} - a_{12}a_{21})((a_{11}a_{22} - a_{21}a_{12})k_{11}k_{22} - (a_{11}a_{22} - a_{21}a_{12})k_{12}^2)$
$= (a_{11}a_{22} - a_{12}a_{21})^2 det A_\xi = (< X, X >< Y, Y > - < X, Y >^2)k.$

On the other hand, using [RTE] one sees that
$< A_\xi X, X >< A_\xi Y, Y > - < A_\xi X, Y >^2 =< \xi, II(X, X) >< \xi, II(Y, Y) >$
$- < \xi, II(X, Y) >^2$
$=<< \xi, II(X.X) > \xi, < \xi, II(Y, Y) > \xi > - << \xi, II(X, Y) > \xi,$
$< \xi, II(X, Y) > \xi >=< II(X, X), II(Y, Y) > - < II(X, Y), II(X, Y) >$
$=< (R - R_S)(X, Y)X, Y >.$

Thus we have proved that the Gaussian curvature k (of S) depends on only the first fundamental form (i.e. [RTE] implies [GTE]). Q.E.D.

§5. Proof of [GTE]

The key point of this section is to recover the Gauss's original idea without using modern Riemannian differential geometry. First let us warm up for calculations. The curvature of a curve C is a limit of ratio of (difference of circle angles) : (length on C) , which is well-known as 1-dimensional case. After on the model of curves, Gauss inductively used analogous definition of curvature k of a surface S in three dimentional Euclidean space as follows. At each point (of S) k can been defined as a limit of ratio of (area on unit sphere) : (area on S). In fact, let (x,y), (x + dx , y + dy), (x + δx, $y+\delta y$)*be a triangular element on a curved surface S* , *and let* (X, Y), $(X+dX, Y+dY)$, $(X+\delta X, Y+\delta Y)$ *be the corresponding elements on unit sphere.*

Then Gauss defined that k = (dX $\delta Y - \delta X dY)/(dx\delta y - \delta x dy)$.*Infact,itfollowsfrom*

$$\begin{pmatrix} dX & \delta X \\ dY & \delta Y \end{pmatrix} = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} \begin{pmatrix} dx & \delta x \\ dy & \delta y \end{pmatrix}$$

that k = $X_x Y_y - Y_x X_y$, *where subindex denotes partial derivative.*

Since S is contained in a three dimensional Euclidean space, we are able to write x = x(p,q), y = y(p,q), z = z(p,q) for two parameters p and q . Let (a, b, c) = (x_p, y_p, z_p), $(a', b', c') = (x_q, y_q, z_q)$, *then*

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} a & a' \\ b & b' \\ c & c' \end{pmatrix} \begin{pmatrix} dp \\ dq \end{pmatrix} .$$

For a point on unit sphere ; X = X(x,y), Y = Y(x,y), Z = Z(x,y) , let us put as follows.

(t,u) = ($Z_x, Z_y), (T, U, V) = (Z_{xx}, Z_{xy}, Z_{yy})$,

$\Delta = \left((bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2\right)^{1/2}$, *then we have*

dZ = t dx + u dy , (X, Y, Z) = $\Delta^{-1}(bc' - cb', ca' - ac', ab' - ba')$ $= (1 + t^2 + u^2)^{-1/2}(t, u, -1)$.

Since $Z^2(1 + t^2 + u^2) = 1, X = -tZ, Y = -uZ$, *we have* $dX = -Zdt - tdZ, dY = -Zdu - udZ, dZ = -Z^3(tdt + udu)$.

Hence dX = - $Z^3(1 + u^2)dt + Z^3tudu, dY = Z^3utdt - Z^3(1 + t^2)du$. *Since*

$$\begin{pmatrix} dt \\ du \end{pmatrix} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} , \quad \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dX \\ dY \end{pmatrix}$$

$$= \begin{pmatrix} -Z^3(1+u^2) & Z^3tu \\ Z^3ut & -Z^3(1+t^2) \end{pmatrix} \begin{pmatrix} dt \\ du \end{pmatrix} ,$$

we obtain that

$$\begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} -Z^3(1+u^2) & Z^3tu \\ Z^3ut & -Z^3(1+t^2) \end{pmatrix} \begin{pmatrix} T & U \\ U & V \end{pmatrix} .$$

Therefore, k = $Z^6(1 + t^2 + u^2)(TV - U^2) = Z^4(TV - U^2)$ $= (TV - U^2)/(1 + t^2 + u^2)^2$.

Furthermore, let us write as follows.
(P, Q, R) = ($W_x, W_y, W_z), (P', Q', R') = (W_{xx}, W_{yy}, W_{zz}), (P'', Q'', R'')$ $= (W_{yz}, W_{xz}, W_{xy})$ *for a function* $W = W(x, y, z)$. *Then*

$$\begin{pmatrix} dP \\ dQ \\ dR \end{pmatrix} = \begin{pmatrix} P' & R'' & Q'' \\ R'' & Q' & P'' \\ Q'' & P'' & R' \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

If dW = P dx + Q dy + R dz = 0 , then P +tR = Q +uR = 0 , because of dz = t dx + u dy . Using dt = d(-P/R) = (-R dP + P dR) / R^2 *and* $dz = (-P/R)dx + (-Q/R)dy$, *one sees that*

$$R^3dt = (-R^2P' + 2PRQ'' - P^2R')dx + (PRP'' + QRQ'' - PQR' - R^2R'')dy,$$

$$R^3du = (PRP'' + QRQ'' - PQR' - R^2R'')dx + (-R^2Q' + 2QRP'' - Q^2R')dy,$$

$$R^3Tdx + R^3Udy = (-R^2P' + 2PRQ'' - P^2R')dx + (PRP'' + QRQ'' - PQR' - R^2R'')dy,$$

$$R^3Udx + R^3Vdy = (PRP'' + QRQ'' - PQR' - R^2R'')dx + (-R^2Q' + 2QRP'' - Q^2R')dy.$$

Thus we conclude that

$$R^3T = -R^2P' + 2PRQ'' - P^2R',$$

$$R^3U = PRP'' + QRQ'' - PQR'' - R^2R'',$$

$$R^3V = -R^2Q' + 2QRP'' - Q^2R'.$$

Since TV-U^2 $= (1+t^2+u^2)^2k = (1+P^2/R^2+Q^2/R^2)^2k = (R^2+P^2+Q^2)^2k/R^4$, *we have* $(R^2+P^2+Q^2)^2k = (R^3T)(R^3V)-(R^3U)^2$
$= P^2(R'Q'-(P'')^2)+Q^2(P'R'-(Q'')^2)+R^2(P'Q'-(R'')^2)+2QR(Q''R''-P'P'')+2PR(P''R''-Q''Q')+2PQ(Q''P''-R'R'')$.

This formula was obtained in the former part of Gauss's paper [G]. In a case of W(x,y,z) = W(x,y) - z = 0, then P = $W_x, Q = W_y, R = W_z = -1, P' = W_{xx}, Q' = W_{yy}, R' = W_{zz} = 0, P'' = W_{yz} = 0, Q'' = W_{xz} = 0$, *and* $R'' = W_{xy}$. *Hence we have*

$(1+W_x^2+W_y^2)^2k = W_{xx}W_{yy}-W_{xy}^2$. *This is nothing but a formula due to G.Monge.*

After that he had changed up to final objective as follows.

Let us recall Gauss's notations ;

$$\begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{pmatrix} = \begin{pmatrix} X_{pp} & X_{qp} & X_{qq} \\ Y_{pp} & Y_{qp} & Y_{qq} \\ Z_{pp} & Z_{qp} & Z_{qq} \end{pmatrix},$$

(A,B,C) = (b c' - c b' , c a' - a c' , a b' - b a') .

Then $\Delta = (A^2+B^2+C^2)^{1/2}, (A,B,C) = (X\Delta, Y\Delta, Z\Delta)$. *It follows from* $(aX+bY+cZ)dp+(a'X+b'Y+c'Z)dq = 0$ *that* $Adx+Bdy+Cdz = (aX\Delta+bY\Delta+cZ\Delta)dp+(a'X\Delta+b'Y\Delta+c'Z\Delta)dq = 0$. *Hence* $tdx+udy = dZ = -A/Cdx-B/Cdy$ *implies that* $Z_x = t = -A/C, Z_y = u = -B/C$. *Since* $Cdp = (ab'-ba')dp = b'adp+b'a'dq-b'a'dq-ba'dp = b'dx-a'dy, Cdq = -bdx+ady$, *we have* $dt = d(-A/C) = -C^{-2}\big((A_pdp+A_qdq)C-A(C_pdp+C_qdq)\big)$.
Hence
$C^3dt = \big(b'(AC_p-CA_p)+b(CA_q-AC_q)\big)dx-\big(a'(AC_p-CA_p)+a(CA_q-AC_q)\big)dy$.

By using permutation of A and B , we obtain similar formula as follows.

$C^3du = \big(b'(BC_p-CB_p)+b(CB_q-BC_q)\big)dx-\big(a'(BC_p-CB_p)+a(CB_q-BC_q)\big)dy$.

By the way, $A_p = \partial(bc' - cb')/\partial p = b_p c' + b(c')_p - c_p b' - c(b')_p$
$= c'y_{pp} + bz_{qp} - cy_{qp} - b'z_{qp}$, $A_q = c'y_{qp} + bz_{qq} - cy_{qq} - b'z_{qp}$,

$B_p = a'z_{pp} + cx_{qp} - az_{qp} - c'x_{pp}$, $B_q = a'z_{qp} + cx_{qq} - az_{qq} - c'x_{qp}$,

$C_p = b'x_{pp} + ay_{qp} - bx_{qp} - a'y_{pp}$, $C_q = b'x_{qp} + ay_{qq} - bx_{qq} - a'y_{qp}$.

Therefore $C^3 dt = \big(b'A(b'x_{pp} + ay_{qp} - bx_{qp} - a'y_{pp}) - b'c'(c'y_{pp} + bz_{qp} - cy_{qp} - b'z_{pp}) + bC(c'y_{qp} + bz_{qq} - cy_{qq} - b'z_{qp}) - bA(b'x_{qp} + ay_{qq} - bx_{qq} - a'y_{qp})\big)dx - \big(a'A(b'x_{pp} + ay_{qp} - bx_{qp} - a'y_{pp}) - a'C(c'y_{pp} + bz_{qp} - cy_{qp} - b'z_{pp}) + aC(c'y_{qp} + bz_{qq} - cy_{qq} - b'z_{qp}) - aA(b'x_{qp} + ay_{qq} - bx_{qq} - a'y_{qp})\big)dy.$

Since $dt = Tdx + Udy, du = Udx + Vdy$, *one sees that* $C^3T = x_{pp}A(b')^2 + y_{pp}B(b')^2 + z_{pp}C(b')^2 - 2x_{qp}Abb' - 2y_{qp}Bbb' - 2z_{qp}Cbb' + x_{qq}Ab^2 + y_{qq}Bb^2 + z_{qq}Cb^2$, $C^3U = -x_{pp}Aa'b' - y_{pp}Ba'b' - z_{pp}Ca'b' + x_{qp}A(ab' + ba') + y_{qp}B(ab' + ba') + z_{qp}C(ab' + ba') - x_{qq}Aab - y_{qq}Bab - z_{qq}Cab$, $C^3V = x_{pp}A(a')^2 + y_{pp}B(a')^2 + z_{pp}C(a')^2 - 2x_{qp}Aaa' - 2y_{qp}Baa' - 2z_{qp}Caa' + x_{qq}Aa^2 + y_{qq}Ba^2 + z_{qq}Ca^2$.

Therefore $C^6(TV - U^2) = C^3T\ C^3V - (C^3U)^2 = ((Ax_{pp} + By_{pp} + Cz_{pp})(Ax_{qq} + By_{qq} + Cz_{qq}) - (Ax_{qp} + By_{qp} + Cz_{qp})^2)\ C^2$ *implies that*

$k\,(A^2 + B^2 + C^2)^2 = (Ax_{pp} + By_{pp} + Cz_{pp})(Ax_{qq} + By_{qq} + Cz_{qq}) - (Ax_{qp} + By_{qp} + Cz_{qp})^2.$

Now write by

$$\begin{pmatrix} E & F & m & m' & m'' \\ F & G & n & n' & n'' \end{pmatrix} = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} \begin{pmatrix} a & a' & x_{pp} & x_{qp} & x_{qq} \\ b & b' & y_{pp} & y_{qp} & y_{qq} \\ c & c' & z_{pp} & z_{qp} & z_{qq} \end{pmatrix}$$

$$\begin{pmatrix} D & D' & D'' \end{pmatrix} = \begin{pmatrix} A & B & C \end{pmatrix} \begin{pmatrix} x_{pp} & x_{qp} & x_{qq} \\ y_{pp} & y_{qp} & y_{qq} \\ z_{pp} & z_{qp} & z_{qq} \end{pmatrix}.$$

Then E G - F^2 = $A^2+B^2+C^2 = \Delta$ *and* $k(A^2+B^2+C^2)^2 = DD'' - (D')^2$.

Recalling that (C,c,c') is orthogonal to (A, b' C - c' B , c B - b C) and that (B,b,b') is orthogonal to (A, b' C - c' B , c B - b C) , we know that ($A^2 + B^2 + C^2)x_{pp} = \big(A(bc' - cb') + a(b'C - c'B) + a'(cB - bC)\big)x_{pp} = D(bc' - cb') + m(b'C - c'B) + n(cB - bC) = AD + m(aG - a'F) + n(a'E - aF) = AD + a(mG - nF) + a'(nE - mF)$.

By similar way (i.e. (A,a) $\longmapsto (B, b), (A, a) \longmapsto (C, c))$, *we know that*

($A^2 + B^2 + C^2)y_{pp} = BD + b(mG - nF) + b'(nE - mF)$,

($A^2 + B^2 + C^2)z_{pp} = CD + c(mG - nF) + c'(nE - mF)$.

Hence ($A^2 + B^2 + C^2)(x_{pp}x_{qq} + y_{pp}y_{qq} + z_{pp}z_{qq}) = (Ax_{qq} + By_{qq} + Cz_{qq})D + (ax_{qq} + by_{qq} + cz_{qq})(mG - nF) + (a'x_{qq} + b'y_{qq} + c'z_{qq})(nE - mF)$
= D'' D + G m'' m - F (m'' n + n'' m) + E n'' n .

It follows from

$$\det\begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix} = A^2 + B^2 + C^2 \neq 0$$

that

$$\begin{pmatrix} x_{qp} \\ y_{qp} \\ z_{qp} \end{pmatrix} = \begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix}^{-1} \begin{pmatrix} D' \\ m' \\ n' \end{pmatrix}.$$

In fact,

$$(A^2+B^2+C^2)\begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix}^{-1} = \begin{pmatrix} A & aG - a'F & a'E - aF \\ B & bG - b'F & b'E - bF \\ C & cG - c'F & c'E - cF \end{pmatrix}$$

Hence $(A^2+B^2+C^2)(x_{qp}^2+y_{qp}^2+z_{qp}^2)$

$= \Big(AD' + (aG - a'F)m' + (a'E - aF)n'\Big)x_{qp} + \Big(BD' + (bG - b'F)m' + (b'E - bF)n'\Big)y_{qp} + \Big(CD' + (cG - c'F)m' + (c'E - cF)n'\Big)z_{qp}$

$= (D')^2 + G(m')^2 - 2Fm'n' + E(n')^2$. *Moreover,* $(A^2+B^2+C^2)(x_{pp}x_{qq} + y_{pp}y_{qq} + z_{pp}z_{qq} - x_{qp}^2 - y_{qp}^2 - z_{qp}^2$

$= D''D - (D')^2 + E(nn'' - (n')^2) + F(2m'n' - m''n - mn'') + G(m''m - (m')^2)$.

Recalling the following definitions ; $E = a^2+b^2+c^2, F = aa'+bb'+cc', G = (a')^2+(b')^2+(c')^2,\ (a,b,c) = (x_p, y_p, z_p),\ (a',b',c') = (x_q, y_q, z_q),$
we have $E_p = 2ax_{pp} + 2by_{pp} + 2cz_{pp} = 2m,\ E_q = 2m', F_p = m' + n, F_q = m'' + n', G_p = 2n', G_q = 2n''$.

Eventually,

$G_{pp} = 2(x_{qp}^2 + x_q x_{qpp} + y_{qp}^2 + y_q y_{qpp} + z_{qp}^2 + z_q z_{qpp}),$

$F_{pq} = x_{ppq}x_q + x_{pp}x_{qq} + (x_{pq})^2 + x_p x_{qpq} + y_{ppq}y_q + y_{pp}y_{qq} + (y_{pq})^2 + y_p y_{qpq} + z_{ppq}z_q + z_{pp}z_{qq} + (z_{pq})^2 + z_p z_{qpq},$

$E_{qq} = 2\Big((x_{pq})^2 + x_p x_{pqq} + (y_{pq})^2 + y_p y_{pqq} + (z_{pq})^2 + z_p z_{pqq}\Big).$

Hence $-1/2\ E_{qq} + F_{qp} - 1/2\ G_{pp} = x_{pp}x_{qq} + y_{pp}y_{qq} + z_{pp}z_{qq} - x_{qp}^2 - y_{qp}^2 - z_{qp}^2$, *then* $(A^2+B^2+C^2)(-1/2\ E_{qq} + F_{qp} - 1/2\ G_{pp}) = k(A^2+B^2+C^2)^2 + E(nn'' - (n')^2) + F(2m'n' - m''n - mn'') + G(m''m - (m')^2)$.

Therefore $4(A^2+B^2+C^2)^2 k = (A^2+B^2+C^2)(-2E_{qq} + 4F_{qp} - 2G_{pp}) + E\Big(G_p - (2F_p - E_q)G_q\Big) + F\Big((2F_q - G_p)(2F_p - E_q) + E_pG_q - 2E_qG_p\Big) + G\Big(E_q^2 - (2F_q - G_p)E_p\Big)$.

Thus we have proved Gauss's Theorema Egregium with explicit form as follows.

$4(EG - F^2)^2 k = -2(EG - F^2)(E_{qq} - 2F_{qp} + G_{pp}) + E(G_p^2 - 2F_pG_q + E_qG_q) + F(E_pG_q - E_qG_p - 2E_qF_q + 4F_pF_q - 2F_pG_p) + G(E_pG_p - 2E_pF_q + E_q^2)$. *Q.E.D.*

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P.S.

The birth of sectional curvature, which is one of the important ingredients in Riemannian geometry, have been found in the section four as follows.

$$(\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2)k = \langle (R - R_S)(X,Y)X, Y\rangle .$$

Since $R(Y,X)Z = -R(X,Y)Z$, $\langle R(X,Y)W, Z\rangle = -\langle R(X,Y)Z, W\rangle$, $\langle R(Y,X)W, Z\rangle = \langle R(X,Y)Z, W\rangle$, it follows from first Bianchi identity

(i.e. $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$) that the Riemannian curvature tensor can be written by sectional curvatures as follows.

$$\begin{aligned}6\langle R(X,Y)Z, W\rangle =& \langle R(X+W,Y+Z)(Y+Z), X+W\rangle - \langle R(X+W,Y)Y, X+W\rangle - \langle R(X+W,Z)Z, X+W\rangle - \langle R(X,Y+Z)(Y+Z), X\rangle - \langle R(W,Y+Z)(Y+Z), W\rangle \\ &+ \langle R(X,Z)Z, X\rangle + \langle R(W,Y)Y, W\rangle - \langle R(Y+W,X+Z)(X+Z), Y+W\rangle + \langle R(Y+W,X)X, Y+W\rangle + \langle R(Y+W,Z)Z, Y+W\rangle + \langle R(Y,X+Z)(X+Z), Y\rangle \\ &+ \langle R(W,X+Z)(X+Z), W\rangle - \langle R(Y,Z)Z, Y\rangle - \langle R(W,X)X, W\rangle .\end{aligned}$$