MATHEMATICAL THEORIES DESCRIBING CAPILLARY ACTION BY LAPLACE, GAUSS AND POISSON
LAPLACE, GAUSS 及び *POISSON* による毛細管現象記述の数学的理論

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Abstract. We discuss the mathematical theory of deduction of the capillary action by Laplace, Gauss, Poisson. These share the common concept of attraction and repulsive force on continuum, which is realized with two constants. The former two are deduce the equations of the capillary surface, and the latter, Poisson confirms the formulae, in another analytical problems. We assert the two constants are used to formulate the equation of the Navier-Stokes equations, due to Laplace' theory of capillarity.

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1. The "two-constant" theory in capillarity

Gauss didn't mention the following fact, and Bowditch [1] also didn't comment on Gauss' work in Laplace's total works [6] except for only one comment of the name "Gauss" [6, p.686]. [2]

N.Bowditch comments as follows :

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat taken into consideration, because he supposed it to be unnecessary. But in 1819[3] he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$ [6, p.685].

In his historical descriptions about the study of capillary action, we would like to recognize that there is no counterattack to Gauss, but the correct valuation. Gauss [2] stated his conclusions about the papers by Laplace as follows : we can not accept the papers by Mr. Laplace ; in p.5, since not only he developed clearly incorrect argument but also showed even the false proofs : we consider that *his calculations in the pages and the following after p.44 are the vain effects.*[4] [2, pp.33-34] (italic and trans. mine.)

2. Laplace papers of the capillary action

2.1. Laplace's conclusions of theory of the capillary action.

Laplace stated his "*complete theory*" of attraction which have an effect on the capillary action

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[1](⇓) The present work is a reprint, in four volumes, of Nathaniel Bowditch's English translation of volumes I, II, III and IV of the French-language treatise *Traité de Mécanique Céleste* by P.S.Laplace. The translation was originally published in Boston in 1829, 1832, 1834, and 1839, under the French title, "*Mécanique Céleste*", which has now been changed to its English-language form, "*Celestial Mechanics.*"

[2](⇓) Bowditch's comment number [9173g].

[3](⇓) Poisson comments this fact in [7, p.19].

[4](⇓) There are 35 pages of calculation between p.44 and p.78 in his *Supplément.*

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in the introduction [3], as follows : From the translation by Bowditch [6], for brevity, we show the corresponding part with above as follows :

> From these results, relative to bodies terminated by sensible segments of a spherical surface, I have deduced this general theorem. *"In all the laws which render the attraction insensible at sensible distance, the action of body terminated by a curve surface, upon an infinitely narrow interior canal, which is perpendicular to that surface, at any point whatever, is equal to the half sum of the actions upon the same canal, of two spheres which have the same radii as the greatest and the least radii of curvature of the surface at that point."* [6, p.689]

2.2. **Laplace's theory of the capillary action.**

Laplace's theories of the capillary action are described in the 14 articles. We cite only the contents of *no* 1 ([4, pp.10-14]) of theory of [4] pointed out by Gauss:
¶ *no* 1 of the theory of capillary action : To have the action of the entire sphere of which the radius is b, let suppose $b - u = z$; this action will be equal to the integral

$$2\pi \int \frac{(b-z)}{b}.dz.\Psi(z),$$

taken from $z = 0$ up to $z = b$. Let hence K the integral $2\pi. \int dz.\Psi(z)$ taken in this limits, and H the integral $2\pi. \int zdz.\Psi(z)$ take in the same limits ; the preceding action will turn into

$$K - \frac{H}{b}.$$

We need to observe here that K and H can be considered as being independent of b ; because $\Psi(z)$ isn't being sensible than of insensible distance, it is indifferent to take the preceding integrals, from $z = 0$ up to $z = b$, or from $z = 0$ up to $z = \infty$; so that we can suppose that K and H respond to these last limits. [4, p.13] (trans. mine.) ($\Downarrow$) This means that

$$K = 2\pi \int \Psi(z), \qquad H = 2\pi \int z\Psi(z)dz \ ;$$

where the limits are from $z = 0$ to $z = b$ or from $z = 0$ to $z = \infty$. These two constants are the original of what we called the two constants, in the 1805's paper [4] by Laplace, so that we think, it is noteworthy. ($\Uparrow$)
¶ *no* 4 ([4, p.18-23]) of the theory of capillary action : Let O (fig. 3) [5] be the lowest point of the surface AOB of the water contained in the tube. Let name z the vertical coordinate OM, x and y the two horizontal coordinates of a certain point N of the surface. Let call R the longest and R' the shortest of contacting radii of the surface at this point. R and R' are the two roots of the equation [6]

$$R^2(rt - s^2) - R\sqrt{(1+p^2+q^2)}\{(1+q^2)r - 2pqs + (1+p^2)t\} + (1+p^2+q^2)^2 = 0, \tag{1}$$

where,

$$p = \frac{dz}{dx} \ ; \quad q = \frac{dz}{dy} \ ; \quad r = \frac{d^2z}{dx^2} \ ; \quad s = \frac{d^2z}{dxdy} =^* \frac{dp}{dy} =^* \frac{dq}{dx} \ ; \quad t = \frac{d^2z}{dy^2}. \tag{2}$$

We will have hence

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2)\frac{dp}{dx} - pq\left(\frac{dp}{dy} + \frac{dq}{dx}\right) + (1+p^2)\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}} = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}} \tag{3}$$

[5]($\Downarrow$) cf. fig. 1.
[6]($\Downarrow$) (1) is a quadratic equation with respect to R.

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Posed thus, if we represent a certain, infinitely long canal NSO, it must hold, with the law of the equilibrium of fluid contained in this canal, b and b' being the longest and shortest of the contacting radii of the surface at the point O and g being the weight.

$$K - \frac{H}{2}\Big(\frac{1}{R} + \frac{1}{R'}\Big) + gz = K - \frac{H}{2}\Big(\frac{1}{b} + \frac{1}{b'}\Big)\ ; \quad \Rightarrow \quad \Big(\frac{1}{R} + \frac{1}{R'}\Big) - \frac{2gz}{H} = \frac{1}{b} + \frac{1}{b'}\ ; \tag{4}$$

In effect, the action of the fluid on the canal at the point N is, with this one precedes, $K - \frac{1}{2}H(\frac{1}{R} + \frac{1}{R'})$, and moreover, the height of the point over the point O is z. The preceding equation gives, in substituting its value for $1/R + 1/R'$, its value, [7]

$$(a) \qquad \frac{(1+q^2).r - 2pqs + (1+p^2).t}{(1+p^2+q^2)^{\frac{3}{2}}} - \frac{2gz}{H} = \frac{1}{b} + \frac{1}{b'};$$

[4, p.19] (trans. mine.) [8]

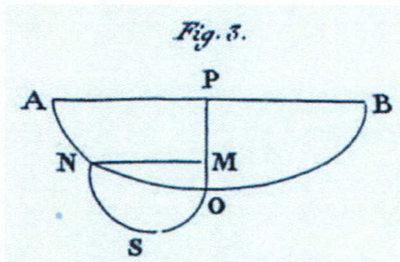

fig.1 a meniscus in a canal.

3. Gauss' papers of the capillary action

Gauss states common motivations with Laplace about MD equations. For example, in §10, §11, §12, which we mention below, he states the difficulties of integral $\int r^2 \varphi r.dr$, in which he confesses that he also is included in the person who feels difficulties to calculate the MD integral.

4. *Principia generalia theoriae figurae fluidrum in statu aequilibrii. (General principles of theory on fluid figure in equilibrium state)*

Gauss introduces his expression of curved surface.

$$\xi = -\zeta.\frac{dz}{dx}, \qquad \eta = -\zeta.\frac{dz}{dy}, \qquad d\zeta = \xi\zeta^2 d\frac{dz}{dx} + \eta\zeta^2 d\frac{dz}{dy} \tag{5}$$

$$\begin{aligned}\frac{d\xi}{dx} &= -\zeta\frac{d^2z}{dx^2} - \frac{dz}{dx}.\frac{d\zeta}{dx} = -\zeta\frac{d^2z}{dx^2} \underbrace{-\zeta\frac{dz}{dx}}_{=\xi}\xi\zeta\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\ &= -\zeta(1-\xi^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta(\eta^2+\zeta^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}\end{aligned}$$

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[7] (⇓) From (3) and (4) we get it.

[8] (⇓) the equation symbol $=^*$ in the expression (2) and $\Rightarrow$ in (4) are by mine.

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TABLE 1. Comparison of Q and V in $\delta U = \int QdP + \int VdU$ between analytic and geometric method

analytic method	geometric method
$Q = \left(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\right)X \quad + \left(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\right)Y$	$Q = -\delta e.\cos(5,7)$
$V = \left(\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\right)\zeta\delta x \quad + \left(\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\right)\zeta\delta y \quad + \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)\zeta\delta z$	$V = \delta e.\cos(4,5).\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$

$$\begin{aligned}
\frac{d\eta}{dy} &= -\zeta\frac{d^2z}{dy^2} + \eta^2\zeta\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta(1-\eta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\
&= -\zeta(\xi^2+\zeta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\
\frac{d\xi}{dx} + \frac{d\eta}{dy} &= -\zeta^3\Big[\frac{d^2z}{dx^2}\Big\{1 + \Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{2d^2z}{dx.dy}.\frac{dz}{dx}.\frac{dz}{dy} + \frac{d^2z}{dy^2}\Big\{1 + \Big(\frac{dz}{dx}\Big)^2\Big\}\Big], \\
\text{where,} \quad \zeta^3 &= \Big[1 + (\frac{dz}{dx})^2 + (\frac{dz}{dy})^2\Big]^{-\frac{3}{2}}. \qquad (6)
\end{aligned}$$

5. Poisson's paper of capillarity

5.1. Poisson's comments on Gauss [1].

Poisson [7] commented in the preface about Gauss [1]:

- Gauss' success is due to the merit of his $\prec$ *characteristic* $\succ$
- even Gauss uses the same method as the given physics by Laplace.
- Gauss calculates by the condition only the same density and incompressibility

5.2. Poisson's two constants : K and H in capillary action.

We cite Poisson's K and H from [7, 12-14].

$$K = 2\pi\rho^2 q \int_0^\infty r^3 \varphi r dr$$

where,

$$q \equiv \int_0^\infty \int_0^\infty \frac{(y+z)dydz}{[1+(y+z)^2]^{\frac{3}{2}}} = \frac{1}{3}\int_0^\infty \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{1}{3}$$

$$(1)_P \quad K = \frac{2}{3}\pi\rho^2 \int_0^\infty r^3\varphi r\, dr \qquad (7)$$

$$\eta = u\sin v, \quad \eta' = u\cos v, \qquad \zeta = Q\eta^2 + Q'(\eta')^2 + Q''\eta\eta'$$

We denote λ and λ' radii of two principle curvatures.

$$\frac{1}{\lambda} = \frac{d^2\zeta}{d\eta^2} = 2Q, \quad \frac{1}{\lambda'} = \frac{d^2\zeta}{d(\eta')^2} = 2Q',$$

The average value

$$\mu = -H(Q+Q') = -\frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big),$$

where, we denote H for convenience sake

$$H \equiv \pi\rho^2 \int_0^\infty \int_0^\infty \varphi r \frac{su^3}{r} duds$$

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where,

$$s = ux, \quad ds = udx, \quad u = \frac{r}{\sqrt{1+x^2}}, \quad du = \frac{dr}{\sqrt{1+x^2}}$$

$$(2)_P \quad H = \pi\rho^2 \int_0^\infty r^4 \varphi r dr \int_0^\infty \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{4}\pi\rho^2 \int_0^\infty r^4 \varphi r \, dr \tag{8}$$

The normal action on this point :

$$(3)_P \quad N = K - \frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big) \tag{9}$$

5.3. Coincidence of Poisson's K and H with Laplace's K and H.

Poisson proved Laplace's formulae as follows :

$$\begin{aligned} K &= \frac{2\pi\rho^2}{3} h^3 \Pi h - \frac{2\pi\rho^2}{3} \int_0^h r^3 \frac{d\Pi r}{dr} dr = \frac{2\pi\rho^2}{3} \int_0^h r^3 \varphi r \, dr, \\ H &= \frac{\pi\rho^2}{4} h^4 \Pi h - \frac{\pi\rho^2}{4} \int_0^h r^4 \frac{d\Pi r}{dr} dr = \frac{\pi\rho^2}{4} \int_0^h r^4 \varphi r \, dr \end{aligned}$$

5.4. Proof by Poisson that the rise in the neighborhood of water surface and wall is due to the abrupt variation of density.

§ 14. Posed thus, call A the liquid contained in a vertical cylinder which has its base on the plane GH and which the generatrix is the straight DL tangent to the wall of the tube, and B the liquid situated around this cylinder and under GH. It goes along one which precedes that the vertical action of the tube and of B on A will independent of the inferior surface of the tube, which the vertical section is represented with $EC'F$, so that we will be capable to replace this surface with a horizontal plane. If we designate then with R the action of B on the part of A situated on this plane, and R' the action of the tube on the part of A situated under this same plane, and if we suppose that the primary force is exercises in the direction of the gravity, and the second in the contrary direction,

$$(7)_3 \qquad 2R' - R = \Delta, \tag{10}$$

for the equilibrium of A.

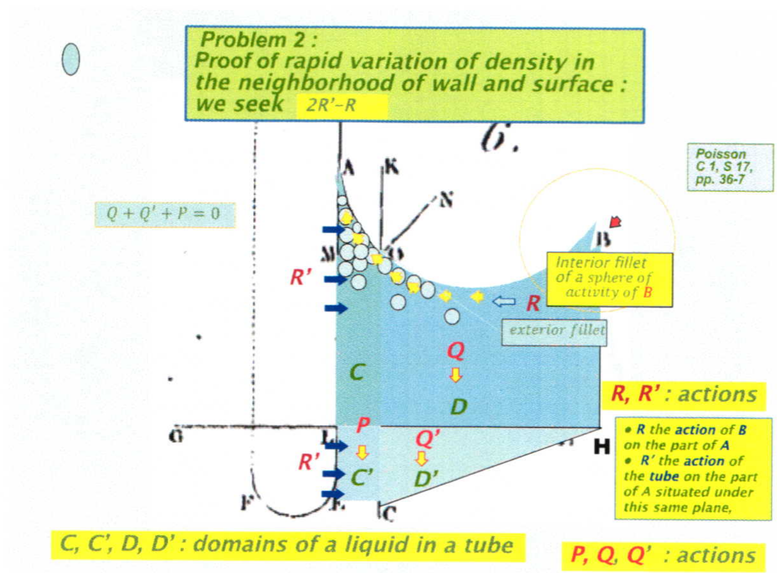

fig.2 the rise in the neighborhood of water surface and wall.

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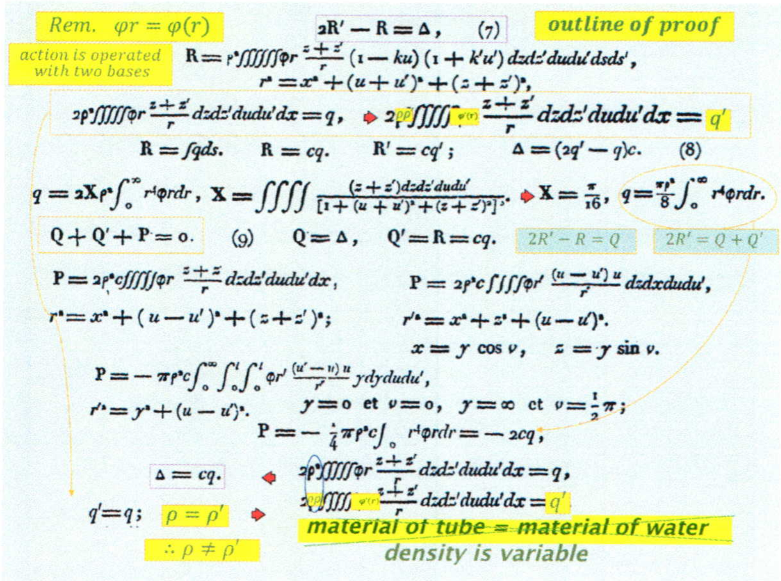

fig.3 Outline of proof by Poisson on the abrupt variation causing the rise in the neighborhood of water surface and wall.

It will rest now to formula the expressions of R and R'. Consequently, were ds an element infinitely small of contour of a ; with the two extremities of ds, trace the planes perpendicular to its direction which is cut along with a vertical passing with the center of the curvature of this contour ; let separate the segment of A composed between these two planes, and fillets infinitely thin with the plane vertical parallel to ds ; and were u the distance from one of these fillets at the plane vertical passing through ds. We will be capable to explain its basis with $(1 - ku)duds$, in supposing this fillet composed in the sphere of activity of B, neglecting the power of u superior to the primary, and designating with k, a constant coefficient which will depend on the curvature of the contour of a, at the point which responds to ds'. The basis of an exterior fillet, belonging to B, which responds to another element ds' of the contour, will be at the same time $(1 + k'u')ds'du'$; u' being the insensible distance from this second fillet to the surface of A, and k' this one which turns k at the point corresponding to ds'. From here, we will conclude without difficulty

$$R = \rho^2 \int\int\int\int\int\int \varphi(r)\frac{z+z'}{r}(1-ku)(1+k'u')dzdz'dudu'dsds',$$

in putting

$$r^2 = x^2 + (u+u')^2 + (z+z')^2.$$

§ 15. (The determination of R and R'.)
designating with $\varphi(r)$ the same function with preceding (no. 2), with x the projection of the arc composed between ds and ds' on the prolongation of ds and with z and z' the perpendicular fallen from a point of A and of a point B on the plane GH, so that r were the distance of a point to the other. At the degree of approximation where we are stayed in all this one which proceeds, we will turn to reduce to the unit the factors $1 - ku$ and $1 + k'u'$. We will be capable next of extending from zero to the infinite, the integrals in respect to u, u', z, z', and integral in respect to x from $x = -\infty$ to $x = +\infty$, namely only from $x = 0$ to $x = \infty$, in doubling the result. In putting

$$2\rho^2 \int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dx \equiv q,$$

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and taking the five integrals from zero to the infinity, we will have then $R = \int qds$.

This last integral will turn to all the points of the contour of a ; and as q won't turn to extend from a point to another, it is followed that if we call c the entire length of this contour, we will have simply $R = cq$. If we designate with $\varphi'(r)$ the mutual attraction of the material of the tube and of that of liquid, relative to the distance r and related at the unit of the volume, ρ', and ρ being the densities of the two materials, and if we represent with q' this one which q turns, when we put $\rho\rho'\varphi'(r)$ instead of $\rho^2\varphi(r)$, we will find similarly $R' = cq'$; by means of this equation (10) will be turn into

$$(8)_3 \qquad \Delta = (2q' - q)c. \tag{11}$$

§ 16. The quintuplicate integral which q represents is reduced easily to a simple integral. In putting at first zx, $z'x$, ux, $u'x$, xdz, xdz', instead of z, z', u and u', and of their primary differentials, the limits zero and the infinity won't changes ; it will result $q = 2X\rho^2 \int_0^\infty r^4\varphi(r)dr$, in putting, to abridge

$$X \equiv \int\int\int\int \frac{(z+z')dzdz'dudu'}{\left[1+(u+u')^2+(z+z')^2\right]^2},$$

finally, we get $X = \frac{\pi}{16}$ and $q = \frac{\pi\rho^2}{8}\int_0^\infty r^4\varphi(r)dr$.

§ 17. (The necessity to regard to the variation of the density of the liquid near the wall of the tube.)

$$(9)_3 \qquad Q + Q' + P = 0, \tag{12}$$

where, $Q = \Delta$, for the equilibrium of this part of the liquid.

The force Q' won't be differ sensibly from the force R of the (no. 14) ; because it would be between them in the ratio of the contour c of the base a to that of the base b, which we can take the one for the other. Therefore, we will have $Q' = R = cq$.

On the force P, its expression will differ from that of R in quintuplicate integral, with the sign of u' and with the limits relative to u and u', namely, that we will have

$$P = 2\rho^2 c\int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dx, \qquad r^2 = x^2 + (u-u')^2 + (z+z')^2 \ ;$$

the integrals relative to x, z, z', being always zero and infinity ; however, those which responds to u and u' isn't extending only from zero to l, in designating with l the length of KL.

$$P = 2\rho^2 c\int\int\int\int\int \varphi(r')\frac{(u-u')u}{r'}dzdxdudu', \qquad (r')^2 = x^2 + z^2 + (u-u')^2.$$

Let again $x = y\cos\nu$, $z = y\sin\nu$. If we substitute these variables y and ν to x and z, it will need to take $dxdz = ydyd\nu$; the limits which respond to ($x = 0$ and $z = 0$) and ($x = \infty$ and $z = \infty$) will be ($y = 0$ and $\nu = 0$), ($y = \infty$ and $\nu = \frac{1}{2}\pi$) ; in effectuating the integration relative to ν, it will result then

$$P = -\pi\rho^2 c\int_0^\infty \int_0^l \int_0^l \varphi(r')\frac{(u-u')u}{r'}ydydudu', \qquad (r')^2 = y^2 + (u-u')^2.$$

Consequently, this triple integral is the same with that which exists in the expression of V of the (no. 8) ; in the analysis of the (no. 9), we will conclude then

$$P = -\frac{1}{4}\pi\rho^2 c\int_0^\infty r^4\varphi(r')dr = -2cq,$$

in neglecting always the term which would have the factor l, and regarding to the value of q of the (no. 16). These values of Q, Q', P, reduce the equation (12) to $\Delta = cq$.

Consequently, for that this value of Δ is accord with that which is given with the equation (11), it might need that it has been $q' = q$; this would cause that the material of the tube would has been the same with that of the liquid. QED.

5.5. **Pressure of liquids, modified with the capillary action.**

§ 80. We will have

$$KMN = K'M'N' = \omega, \qquad KMH = K'M'H = i\ ;$$

ω being the angle relative to the material of liquid and with the surface of corps, given with the experience, and obtuse or acute, according as the liquid is elevated or is fallen ; and i designing the same angle with in the precedent number.

I will call Γ the liquid layer adjacent to the surface of corps, and which the section is terminated, otherwise, at the curve MGM', at the normals MN and $M'N'$, and at the portions of curves AN and $A'N'$. I will name L the rest of liquid, and I go to calculate the vertical action of L on the layer Γ, at which I have given the form necessary for that this force can explain by means of quantity which will be given in each case.

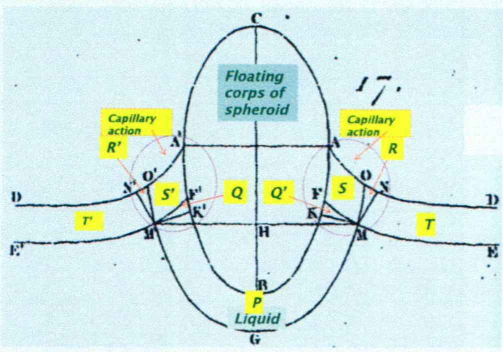

fig.4 Pressure of liquids, modified with the capillary action on a spheroid.

To abridge, I will indicate each party of L or of Γ, consequently, generally each part of liquid with the part of the figure to which it responds. Being thus, the action of $DOGO'D'$ on $KMGM'K'$ isn't other thing with the force N of (no. 76), decomposed vertically and applied to all the elements of the part of surface of Γ which responds to the curve MGM' ; I will represent with P, in supposing oriented in direction contrary of the gravity. I will designate, following this direction, with Q the action of same liquid on the part of Γ which responds to FMK or $F'M'K'$, on its part $FMGM'F'$. It is evident that to have action of L on this last part of Γ, it will need to abolish R from $P+Q$. The action of Γ on the surplus of Γ, namely, on the part corresponding to $NMFA$ or $N'M'F'A'$, will be composed from the action of $EMGM'E'$, which we will represent with S, and the action of the superficial layer or corresponding to $DNME$ or $D'N'M'E'$, which I will designate with T ; the both of one and other oriented in contrary sense of the gravity. The total action of L on Γ will then will be

$$P + Q - R + S + T\ ; \tag{13}$$

which it needs to calculate essentially the five parts which it is composed.

§ 81. (Calculation of pressure)

If we call t the distance of a point arbitrary of the curve MGM' to the axis GC, the horizontal projection of a zone infinitely small of the surface generated with this curve, will be $2\pi t dt$, and the component vertical of the force normal N, applied to all this zone will have for value $2\pi N t dt$; in consequence, we will have

$$P = 2\pi \int_0^r N t dt,$$

in taking r for the value of HM, this one which we can make without sensible error. The part of P which responds to second term of M is the integral $\int Z ds$, which the value will be $2\pi r q \cos i$,

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owing to the (no. 79). In putting for the primary term p of N, its value $c - \rho gz$, we will have then

$$P = \pi cr^2 - 2\pi g\rho \int_0^r ztdt + 2\pi rq\cos i.$$

I will call V the part of volume of corps which is situated downward of plane of the x and y, and which responds, consequently, to the values negative of z ; I will designate with v the part of its volume composed between this plane of the section horizontal of corps, at which the liquid is blocked and which we can, without sensible error, makes pass through the points M and M', instead of A and A'. Let be also k the distance of this section at the plane of the x and y ; in regarding k and v as the quantities positive or negative, according as the point A and A' will be upward or downward of this plane, we will have

$$2\pi \int_0^r ztdt = \pi kr^2 - v - V.$$

If the liquid is extended infinitely around of corps, its surface will be sensibly plane to a certain distance ; in taking this plane for this of the x and y, Y will be the volume of the corps situated below level natural of liquid, and $V + v$ the volume of this corps in contact with the liquid. In this same case will have $c = \Pi$; however, for more generality, I will put

$$c = \Pi + g\rho b;$$

b being a constant which will be null in the case of a liquid indefinite, and which the value will depend on the volume of liquid, when it will have a measure given. From this manner, we will have

$$P = \pi r^2\Pi + \pi g\rho(b - k)r^2 + g\rho(v + V) + 2\pi rq\cos i.$$

If we decompose into elements infinitely small, the part of Γ which responds to $NMFA$, the action of the layer superficial $DNME$ on an element which the thickness is ε, will be the force $U\varepsilon$ of the (no. 41), perpendicular to MN and traced from outward into inward of the element ; we will have then the part of T which responds to this element, in multiplying $U\varepsilon$ with the sine of the angle which makes the straight MN with the vertical traced from bottom to height the point M, of which the angle is equal to HMN, less than the right angle, or to $i + \omega - \frac{1}{2}\pi$; and as we have found $U = -q_\iota$, this part of T will be $q_\iota \cos(i + \omega)$; consequently, this force being the same for all the elements, we will conclude from this, the total value of T, in replacing ε with the circumference $2\pi r$; this which puts

$$T = 2\pi rq_\iota \cos(i + \omega).$$

Each of the forces Q, R S, is deduced similarly from the force $Z\varepsilon$ of the (no. 42), in determining suitably the angles a, b, a', b', and replacing ε with $2\pi r$. Let be, for this, (fig. 18), IMI' a vertical, HM a horizontal, MK and MN of the straights which make the angles i and $i + \omega$ with MH, OMG and FME, the straights perpendicular to MK and MN. We will take the straight IMI' for the axis DCG of the (fig. 12), from which the angles a, b, a', b', are regarded ; and the force $Z\varepsilon$ will be traced along with MI. To deduce Q, it will need to make coincide the lines CA, CB, CA', CB', of the (fig. 12), with the straights MG, MO, MK, MF, of the (fig. 18) ; being thus, we will have

$$Q = 4\pi qr \sin i \cos\omega.$$

Releatively to the force R, we will make coincide the lines CA, CB, CA' , CB', of the (fig. 12), with the lines MN, MO, MG, MF, of the (fig. 18) ; and it will result

$$R = 2\pi qr\Big[\sin i\tan\Big(\frac{1}{4}\pi - \frac{1}{2}\omega\Big) - \sin(i + \omega)(1 - \tan\frac{1}{2}\omega)\Big].$$

Finally, to regard of the force S, we will make coincide the lines CA, CB,CA', CB', of the (fig. 12), with the lines MG, ME, MF, MN, of the (fig. 18) ; this one which will require that we takes and from here we will conclude

$$S = 2\pi qr\Big[\sin i \tan\Big(\frac{1}{4}\pi - \frac{1}{2}\omega\Big) - \sin i \cot\frac{1}{2}\omega - \sin(i+\omega)\Big].$$

By means of these values of Q, R, S, we will have

$$Q - R + S = 2\pi qr\Big[2\sin i\cos\omega - \sin(i+\omega)\tan\frac{1}{2}\omega - \sin i\cot\frac{1}{2}\omega\Big].$$

equation which we can put also :

$$Q - R + S + 2\pi qr\cos i = 2\pi qr\cos(i+\omega).$$

Hence, owing to the values of P and T, the total pressure exercised on the floating corps, in sense contrary of the gravity, will have for expression

$$\pi r^2\Pi + \pi g\rho br^2 + g\rho V - q\rho\Big[\pi kr^2 - v - \pi ra^2\cos(i+\omega)\Big],$$

in making, as in the precedent chapter,[9]

$$q + q_{\imath} = \frac{1}{2}H, \qquad H = g\rho a^2, \tag{14}$$

We recall that ω at the same signification with in this chapter, and that $i+\omega$ is the angle composed between the radius of corps which the length is r, and the normal exterior of liquid, traced with the extremity of this radius, of which responds to the section of corps where the liquid is blocked.

§ 84. (To solve one of question the most interesting of the theory of the capillary action.)

To determine the effect of the capillarity on the horizontal pressures, I will suppose that the floating corps were composed between two planes vertical and parallel, of one very large, so that can neglect without sensible error, the part of the pressure which hold near their extremity, relatively to the total pressure, and consider the around of the liquid and the pressure as constants in all the length of each plane. The corps will be terminated, in height and in base, with the certain surface ; will will suppose the inferior surface entirely immersed,, and the surface superior entirely outward of the liquid, The (fig. 19) represents a section of this corps vertical and perpendicular to these two lateral faces. The figures AD and $A'D'$ were the sections of the surface of liquid, of part and another corps, which cuts their two faces at the point A and A'. These curves are different, and A and A' are not belonged with a same horizontal straight. According as each of these points is upward or downward of the level of liquid, the curve corresponds will turn its concavity with in height or with in low. The straight LL' is the intersection of the plane of the figure and of a horizontal, which I will take for that of the x and y, and that I will suppose to a distance h under the level of liquid. It cuts the two faces of liquid at the point C and C', situated upward of the part curve of corps and downward of A and A'. I will put

$$AC \equiv h + k, \qquad A'C' \equiv h + k_{\imath}\ ;$$

k and $k_{\imath}$ being the quantity positive or negative, according as A and A' are upward or downward of the level of liquid.

Posed thus, the horizontal pressures are canceled out on the part of corps situated under plane of the x and y ; those which provides from the atmospheric pressure is canceled equally on the entire corps. Upward of the points C and C', the radii of couvature λ and λ' being infinite, the normal pressure N will reduced to its part p, which we will be capable to represent the value with

$$p = g\rho(h - z),$$

[9]($\Downarrow$) The supposition of $F = bH$ and $H = g\rho a^2$ are defined in (no. 75).

without consideration of Π. The horizontal pressures which provide this force p, and which hold on the part of corps corresponded to CA and CA', will be hence

$$g\rho l\int_0^{h+k}(h-z)dz, \qquad g\rho l\int_0^{h+k_\imath}(h-z)dz,$$

in designating with l the measure of corps, and supposing the same for the two surfaces. In consequence, if we effectuate the integrations and if we call δ the excess of the pressure of the force p, which would push the corps in everywhere, we will have

$$\delta = \frac{1}{2}g\rho l(k^2 - k_\imath^2). \tag{15}$$

But, the quantity p isn't the pressure of the liquid in all its height ; it ceases to exist at a distance from the surface less than the radius of the molecular activity ; and although this holds only in an insensible thickness, the pressure exercised with the superficial layer of the liquid isn't less a sensible quantity, which it isn't permitted to neglect.

§ 85. Let then M a point of liquid situated at right of the figure, at the distances of AC and AD, less than the radius activity of the tube and of liquid. With this point, trace a vertical OMG which cuts AD at the point O, a perpendicular to this curve AD which meets at the point N, a horizontal MH which meets AC at the point K, and a curve FME parallel to AND. We will have to determine the components horizontal of the same fores which we have previously considered (no. 80) the vertical components Q, R, S, T. I will designate with Q', R', S', T' ; the value of each of these quantities will have l for factor ; and, without consideration of this factor, T will be the component along with MK of the force U of the (no. 41), which operates along with MF, and which the value is $-q_\imath$; from here, we conclude

$$T = -q_\imath l\sin\omega \;;$$

ω being always the angle given $\angle KMN$. On the values of Q', R', S', they are obtained, as those of Q, R, S, owing to the formulae (16) or (17) of the (no. 44) ;[10] where, instead of regarding the angles a, b, a', b', from the vertical IMI' (fig. 18), it will need to give them for origin the horizontal MH or its prolongation MH', and make coincide the straights MK and MH, consequently suppose $i=0$.
we will deduce

$$S' = ql\Big[\tan\Big(\frac{1}{4}\pi - \frac{1}{2}\omega\Big) - \cos\omega - \cot\frac{1}{2}\omega\Big].$$

It results from here that the horizontal pressure exercised on the face of corps which responds to AC, to be accurate, on the liquid layer adjacent to this face, will turn to be augmented from a force $Q' - R' + S' + T'$, which the value will be

$$\underbrace{ql\Big(2\cot\omega - \cos\frac{1}{2}\omega - \cot\frac{1}{2}\omega\Big)}_{Q'-R'+S'}\underbrace{-q_\imath l\sin\omega}_{T'}\;;$$

quantity which we can reduce to [11]

$$-(q+q_\imath)l\sin\omega.$$

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[10](⇓)

$$(4)_3\quad Z = q\Big[\sin b'\Big\{\tan\frac{1}{2}(a+b') - \tan\frac{1}{2}(b+b')\Big\} - \sin a'\Big\{\tan\frac{1}{2}(a+a') - \tan\frac{1}{2}(a'+b)\Big\}\Big]. \tag{16}$$

$$(5)_3\quad Z = q\Big[\sin b\Big\{\tan\frac{1}{2}(a'+b) - \tan\frac{1}{2}(b+b')\Big\} - \sin a\Big\{\tan\frac{1}{2}(a+a') - \tan\frac{1}{2}(a+b')\Big\}\Big]. \tag{17}$$

[11](⇓) Using,

$$\sin\alpha = \tan\frac{\alpha}{2}(1+\cos\alpha), \quad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}.$$

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The force which it will turn to augment the pressure relative to the face corresponding to $A'C$, will be similarly

$$-(q+q_\imath)l\sin\omega_\imath.$$

$\omega_\imath$ designating this one which the angle ω turns in respect to this second face of corps, which may not be the same nature with the former. These two forces activate in contrary sense with each other ; and if we call ε the complete value of the excess of the horizontal pressure which pushes the corps in everywhere, we will have

$$\varepsilon=\delta+\big(q+q_\imath\big)l\big(\sin\omega_\imath-\sin\omega_{\imath}.\big),$$

consequently, this one which is the same thing,

$$(9)_5 \qquad \varepsilon=\frac{1}{2}gl\Big[k^2-k_\imath^2+a^2\big(\sin\omega_\imath-\sin\omega_{\imath}.\big)\Big], \tag{18}$$

in regarding to the value of the part δ, [12] and observing that $q+q_\imath=\frac{1}{2}g\;\rho\;a^2$. [13]

This result differs from that of the *Mécanique céleste*, in this one which the author doesn't regard the particular pressure which holds near the surface of the liquid, and which doesn't disappear from the exact value of ε which in the particular case where the two angles ω and $\omega_\imath$ are equal or makes supplementary angle in each other. [14]

6. **Conclusions**

The formulae deduced by Laplace and Gauss are identical, Poisson uses as a commonly known formula. Poisson emphasizes the variation of density in the neighbor of wall and surface, by which the fall or elevation occurred. Today's common knowledge teaches it to us by means of the surface tension, of which Poisson doesn't tell at all, however, the difference between capillarity and surface tension is vague. For example, capillary wave means the wave of surface tension. We can replace a part of action which Poisson uses with surface tension. By the way, the ward 'surface tension' is used already by a Prandtl's textbook in 1933 [8].

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[12] (⇓) cf. The expression (15).

[13] (⇓) cf. The expression (14).

[14] The reasoning which we find at the beginning of the page 43 of the *Théorie de l'Action capillaire*, shows that Laplace has judged this pressure completely negligible, because it responds only to an extent insensible of the surface of corps.

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