Schur の単位円盤上の有界冪級数に関する結果と純粋数学以外への応用

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Schur の複素単位円盤上の有界冪級数に関する論文は，次の３編がある：

[S29] J. Schur (= I. Schur), Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. I, *J. für d. reine u. angew. Math.*, **147**(1917), 205–232 (§§ 1–8),

[S30] J. Schur, —. II, *ibid*, **148**(1918), 122–145 (§§ 9–15),

[S56] I. Schur und G. Szegö, Über die Abschnitte einer im Einheitskreise beschräbkten Potenzreihen, *Sitzungsberichte d. Preuss. Akad. d. Wissenschaften*, **1925**, 545–560.

また，[S29] で引用している自己論文は [S21] であり，[S30] で引用しているのは [S19] である：

[S19] J. Schur, Bemerkungen zur Thorie der beschränkten Bilinearformen mit unentlich vielen Veränderlichen, *J. für d. reine u. angew. Math.*, **140** (1911), 1–28.

[S21] I. Schur, Über einen Satz von C. Carathéodory, Berliner Berichte **1912**, ???–???.

一方，Thomas Kailath (Stanford University) の論文

[Kai] T. Kailath, A theorem of I. Schur and its impact on modern signal processing, in *Operator Theory: Advances and Applications, I.Schur Methods in Operator Theory and Signal Processing,* Vol.**18** (1986), pp.9-30.

では，論文 [S29]–[S30] の結果の Signal Processing 等への応用について論じている．私はこれまでは，Schur の業績の内でも（大なり小なり）表現論と関係のある論文について調査報告してきたが，今回は論文 [Kai] の **abstract** にある惹句に惹かれて全く毛色の違った内容の論文 [S29]–[S30] を調べて報告する．

I. 論文 [Kai] A theorem of I. Schur and ... の要点

1　論文 Schur [S29], [S30] からの応用への可能性

(From **abstract** of [Kai])　As algorithm of Schur for characterizing power series that are bounded in the unit circle is

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shown to have application to a variety of problems in science and engineering. These include speech analysis and synthesis, inverse scattering, decoding of error-correcting codes, synthesis of digital filters, modeling of random signals, Padé approximation for linear systems, and zero location of polynomials.

We also demonstrate relation between Schur's algorithm and the recently introduced concept of *displacement structure* which is fundamental eigenanalysis of matrices. It also has meaningful links with the theory of operators close to Hermitian and close to unitary.

... (2 lines omitted) ...

(From **1. INTRODUCTION**) The name Schur is associated with many terms and concepts that are now widely used in many fields of engineering, e.g., Schur convexity, Schur (Hadamard) matrix products , Schur canonical form for matrices, Schur-Cohn tests for root distribution of polynomials, etc.

⋯ (many lines omitted) ⋯

⋯ ⋯ Of course, this is just another fascinating illustration of the mutually further interaction between mathematics and applications. What may be unique about Issai Schur is that similar accounts could be presented for so many of his papers!

⋯ (several lines omitted) ⋯

第2節以降の目次

2 A pararell algorithm for Toeplitz equations

ここでは，いわゆる speach synthesis の実例を用いて説明されているが，突き詰めると以下のような 'Toeplits linear equations' を解く計算問題になる．論文 [Kai] の用語等を半解のままなので，いきなり結論に突っ走ることとする．m-th order autoagressive linear filter とは，input time sequence $\{e_t; t=0,1,2,\ldots\}$ を入れると，線形の規則で変換して，実数値の output sequence $\{y_t; t=0,1,2,\ldots,\}$ を出力する仕掛けである．output は，expectation を E とすると，

(I.1) $$Ey_t = 0,\ Ey_ty_s = c_{|t-s|},$$

の形と仮定する．この output covariances $c_0, c_1, \ldots, c_m$ を計算したい．$c_0 = 1$ と正規化しておく．与えられた filter coefficients

$((a_{i,1}, a_{i,2}, \ldots, a_{i,m}); \sigma_i^2)$ $(i = 0,1,2,\ldots,m)$，に対して，方程式

(I.2) $$\begin{pmatrix} a_{0,m} & a_{0,m-1} & \cdots & a_{0,m} & 1 \\ a_{1,m} & a_{1,m-1} & \cdots & a_{1,m} & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ a_{m,m} & a_{m,m-1} & \cdots & a_{m,m} & 1 \end{pmatrix} T_m = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & \sigma_0^2 \\ 0 & 0 & \cdots & \cdots & 0 & \sigma_1^2 \\ 0 & 0 & \cdots & \cdots & 0 & \sigma_2^2 \\ 0 & 0 & \cdots & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & \sigma_m^2 \end{pmatrix}$$

ここに，$T_m := \begin{pmatrix} 1 & c_1 & c_2 & \cdots & \cdot & c_m \\ c_1 & 1 & c_1 & \cdots & \cdot & c_{m-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ c_m & c_{m-1} & \cdot & \cdots & c_1 & 1 \end{pmatrix}$ (c_i 未知数)，

により，output covariance $(c_1, c_2, \ldots, c_m)$ を求める．行列 T_m は Toeplitz matrix と呼ばれる．この Toeplits linear equations (I.2) は未知数 $c_1, \ldots, c_m$ に関する線形一次方程式である．単純な加法と乗法という基本計算の回数を数えると，並列計算をしないままだと $O(m^3)$ 回，並列計算ではそれぞれ $O(m^2)$ 回，うまくやっても $O(m \log m)$ 回，必要である．

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第 1 重要点. そこに，Schur [S29] で導入されたいわゆる **Schur Algorithm**[1] を用いれば，$O(m)$ 回の（並列）計算量にまで減らすことが出来る.

第 2 重要点. 実 Hermite 行列 T_m は正定値である.

3 The Schur algorithm

Toeplitz 線形方程式 (I.2) の T_m を決める $c_1, c_2, \ldots, c_m$ に対して，複素領域 $|z| > 1$ における関数を次のようにおく：

$$C(z) := 1 + 2\sum_{i=1}^{m} c_i z^{-i} + O(z^{-m-1}) \tag{I.7}$$

すると，上の**第 2 重要点**が次のような必要十分条件であらわされる：

行列 T_m が正定値 $\overset{\text{iff}}{\Longleftrightarrow}$

必要十分条件 1. $\begin{cases} C(z) \text{ は } |z| > 1 \text{ で解析的,} \\ |z| > 1 \text{ で } \Re\{C(z)\} > 0. \end{cases}$

この段階で生きてくるのが，Schur [S29, Satz XI, §8, p.229]（後述）であって，それが [S29] からの第 2 の貢献である．複素平面上の分数変換 $w' := \dfrac{w-1}{w+1}$ は右半平面 $\Re w > 0$ を単位円盤 $|w'| < 1$ に写す．そこで，

$$S(z) := \frac{C(z) - 1}{C(z) + 1}$$

とおくと，上の必要十分条件は，次に翻訳される：

必要十分条件 2. $\begin{cases} S(z) \text{ は } |z| > 1 \text{ で解析的,} \\ |z| > 1 \text{ で } |S(z)| < 1. \end{cases}$

[1] 単位円盤内で正則な $f(z)$ で，$M(f) := \sup_{|x|<1} |f(x)| = 1$ となるものを連分数型に展開する各 step での有理式の計算公式を 2×2 型の線形変換の公式に書いたもの.

(network theory では，関数 $S(z)$ を scattering functions とよぶ.）Schur による逐次計算法で

$$S_0(z) := S(z),\quad S_{i+1}(z) = \frac{S_i(z) - S_i(\infty)}{z^{-1}\bigl(1 - S_i(\infty)S_i(z)\bigr)} \quad (i = 0, 1, \ldots, m-1),$$

とおくと，Schur の Satz XI を応用して，必要十分条件 2 を書き直すと，

必要十分条件 3. T_m 正定値 $\overset{\text{iff}}{\Longleftrightarrow}$ $|S_i(\infty)| < 1 \ (i = 0, 1, \ldots, m)$.

さらに，$S_i(\infty)$ の逐次計算式は非線形だが，これを 2×2 型の線形変換式に書き直せるので計算量が減小し，上述のように応用面で有用となる.

II. [S29], [S30] J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. I, II

[S29]–[S30] の Contents :

[S29] J. Schur (= I. Schur), Uber Potenzreihen, die im Innern des Einheitskreises beschränkt sind. I.

[S30] J. Schur, —. II.

4)

1. Introduction of continued fraction type algorithm

$w = f(x)$ analytic on $D := \{x \in \boldsymbol{C}\,;\ |x| < 1\}$,

$$M(f) := \sup_{|x|<1} |f(x)|.$$

(II.1) $$\mathfrak{E} := \{f\,;\ M(f) \leq 1\}.$$

Schwarz Lemma (一部). (Hermann A. Schwarz) $M(f) = M(xf)$.

$\alpha \in \boldsymbol{C}$, $|\alpha| < 1$, に対し, $w' := \dfrac{w-\alpha}{1-\overline{\alpha}w}$, $|w| = 1 \Leftrightarrow |w'| = 1$. は,
閉円盤 Cl(D) 上の変換で, $\alpha \to 0$, $0 \to -\alpha$.
$f \in \mathfrak{E}$ に対して, $g := \dfrac{f-\alpha}{1-\overline{\alpha}f} \in \mathfrak{E}$. $M(f) = 1 \Leftrightarrow M(g) = 1$.

(II.2) $$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots \in \mathfrak{E} \quad (x \in D),$$

$|c_0| = 1 \implies f(x) \equiv c_0$ (定数).

基本補題. *For any* $\nu \geq 0$, $|c_\nu| \leq M(f)$. □

$|c_0| < 1$ の場合, $\gamma_0 = c_0$ と置いて,

$$f_1 := \frac{1}{x}\frac{f-\gamma_0}{1-\overline{\gamma_0}f} = \frac{c_1 + c_2 x + c_3 x^2 + \cdots}{(1-\overline{\gamma_0}\gamma_0) - \overline{\gamma_0}c_1 x - \overline{\gamma_0}c_2 x^2 - \cdots} \in \mathfrak{E},$$

$M(f_1) = 1 \iff M(f) = 1$.

$\gamma_1 := f_1(0) = \dfrac{c_1}{1-\overline{c_0}c_0}$, $|\gamma_1| \leq 1$,

- If $|\gamma_1| = 1 \implies f_1 \equiv \gamma_1$,
- If $|\gamma_1| < 1 \implies f_2 := \dfrac{1}{x}\dfrac{f_1-\gamma_1}{1-\overline{\gamma_1}f_1}$, $\gamma_2 := f_2(0)$.

(Schur algorithm) Finite or infinite series of

(II.3) $$f_0 = f,\ f_1,\ f_2,\ f_3,\ \ldots,$$

(II.4) $$f_{\nu+1} = \frac{1}{x}\frac{f_\nu - \gamma_\nu}{1-\overline{\gamma_\nu}f_\nu}, \quad f_\nu = \frac{xf_{\nu+1}+\gamma_\nu}{1+\overline{\gamma_\nu}\,xf_{\nu+1}}, \quad \gamma_\nu = f_\nu(0).$$

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$$M(f_\nu) = 1 \iff M(f) = 1,$$

- $|\gamma_\nu| < 1$ とすると，$|f_\nu(x)| < 1 \iff |f_{\nu+1}(x)| < 1 \quad (|x| < 1)$.
- If $f_\nu \equiv \gamma_\nu \implies f(x) := [x; \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_\nu]$ a rational function,
- Otherwise, $\{f_\nu\}_{\nu \geqslant 0}$ adjoint functions of f,

  $\{\gamma_\nu\}_{\nu \geqslant 0}$ parameter belonging to f.

Case 1. $\{f_\nu\}_{\nu \geqslant 0}$ が無限項. 　このとき，$\forall \nu$, $|\gamma_\nu| < 1$.
とくに，$f_\nu \equiv \gamma_\nu$ のとき，$f_\lambda(x) \equiv \gamma_\lambda = 0 \ (\forall \lambda > \nu)$.
Case 2. $\exists n$ such that $|\gamma_\nu| < 1 \ (0 \le \nu < n)$, $|\gamma_n| = 1$.
このとき，$f(x) = [x; \gamma_0, \gamma_1, \ldots, \gamma_n]$ と書く.

Case 2 occurs if and only if $f(x)$ is a rational function of the form

$$f(x) = \varepsilon \prod_{\nu=1}^{n} \frac{x + \omega_\nu}{1 + \overline{\omega_\nu} x}, \quad 0 \le |\omega_\nu| < 1, \ |\varepsilon| = 1,$$

or

$$f(x) = \varepsilon \frac{x^n + \overline{k_1} x^{n-1} + \cdots + \overline{k_n}}{1 + k_1 x + \cdots + k_n x^n} = \varepsilon \frac{x^n \, \overline{P}(x^{-1})}{P(x)},$$

where $P(x)$ is of degree at most n and becomes zero outside of $\mathrm{Cl}(D)$ (or $\equiv 1$).

2. Functions Φ and Ψ

$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$,

$f_\nu(x) = c_{\nu 0} + c_{\nu 1} x + c_{\nu 2} x^2 + \cdots, \qquad (c_{\lambda 0} = c_\lambda)$,

$c_{\nu\lambda}$ is a rational function of $c_0, \overline{c_0}, c_1, \overline{c_1}, \ldots c_{\nu-1}, \overline{c_{\nu-1}}, c_\nu, c_{\nu+1}, \ldots, c_{\nu+\lambda}$.

(Schur parameters)

$\gamma_\nu := c_{\nu 0} := \Phi(c_0, c_1, \ldots, c_\nu) =: \Phi_\nu$,

a rational function of $c_0, \overline{c_0}, c_1, \overline{c_1}, \ldots c_{\nu-1}, \overline{c_{\nu-1}}, c_\nu$.

$c_\nu = \Psi(\gamma_0, \gamma_1, \ldots, \gamma_\nu) =: \Psi_\nu$

a polynomial function of $\gamma_0, \overline{\gamma_0}, \gamma_1, \overline{\gamma_1}, \ldots \gamma_{\nu-1}, \overline{\gamma_{\nu-1}}, \gamma_\nu$,

$$\Psi_0 = \gamma_0, \quad \Psi_1 = \gamma_1(1 - \overline{\gamma_0}\gamma_0), \quad \Psi_2 = \gamma_2(1 - \overline{\gamma_0}\gamma_0)(1 - \overline{\gamma_1}\gamma_1) - \overline{\gamma_0}{\gamma_1}^2(1 - \overline{\gamma_0}\gamma_0).$$

From $f(1 + \overline{\gamma_0}\, x f_1) = \gamma_0 + x f_1$, and so on, we get

$$\Psi(\gamma_0, \gamma_1, \ldots, \gamma_\nu) = (1 - \overline{\gamma_0}\gamma_0)\Psi(\gamma_1, \gamma_2, \ldots, \gamma_\nu) - \overline{\gamma_0} \sum_{\lambda=0}^{\nu-1} \Psi(\gamma_0, \gamma_1, \ldots, \gamma_\lambda)\Psi(\gamma_1, \gamma_2, \ldots, \gamma_{\nu-\lambda}).$$

上式より，次の逐次方程式を得る：

(A) $$\Psi(\gamma_0, \gamma_1, \ldots, \gamma_\nu) = \gamma_\nu \prod_{\lambda=0}^{\nu-1}(1 - \overline{\gamma_\lambda}\gamma_\lambda) + \Psi',$$

where Ψ' depends only on $\gamma_0, \overline{\gamma_0}, \gamma_1, \overline{\gamma_1}, \ldots \gamma_{\nu-1}, \overline{\gamma_{\nu-1}}$.

In general, $\varphi_\nu(x) := [x; \gamma_0, \gamma_1, \ldots, \gamma_\nu]$ can be given for any value of γ_λ and has the recursion formula

(II.5) $$[x; \gamma_0, \gamma_1, \ldots, \gamma_\nu] = \frac{\gamma_0 + x\,[x; \gamma_1, \gamma_2, \ldots, \gamma_\nu]}{1 + \overline{\gamma_0}\, x\,[x; \gamma_1, \gamma_2, \ldots, \gamma_\nu]},$$

$$[x; \gamma_1, \ldots, \gamma_\nu] = \frac{\gamma_1 + x\,[x; \gamma_2, \gamma_3, \ldots, \gamma_\nu]}{1 + \overline{\gamma_1}\, x\,[x; \gamma_2, \gamma_3, \ldots, \gamma_\nu]}, \quad \ldots, \quad [x; \gamma_\nu] = \gamma_\nu .$$

3. Criterion for the coefficients of a bounded power series

Satz I. *Let* $\gamma_0, \gamma_1, \gamma_2, \ldots$ *be* $|\gamma_\nu| < 1$ $(\forall \nu \geq 0)$. *Then the series*

$$f(x) := \sum_{\nu=0}^{\infty} \Psi(\gamma_0, \gamma_1, \ldots, \gamma_\nu)\, x^\nu = \sum_{\nu=0}^{\infty} c_\nu\, x^\nu =: [x; \gamma_0, \gamma_1, \gamma_2, \ldots]$$

converges for $|x| < 1$, *and* $M(f) \leq 1$. *Moreover for* $|x| < 1$

$$[x; \gamma_0, \gamma_1, \gamma_2, \ldots] = \lim_{\nu\to\infty} [x; \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_\nu]$$

and the convergence is compact uniform.

(Added) $|c_\nu| < 1$ $(\forall \nu \geq 0)$.

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例 (§15 より).

$$\frac{1+x}{2} = \left[x; \frac{1}{2}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \dots\right], \qquad \frac{1}{2-x} = \left[x; \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right],$$

Satz II. (必要十分条件) *The power series*

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

is convergent and $M(f) \leq 1$ *if and only if, for* $\gamma_\nu = \Phi(c_0, c_1, \dots, c_\nu)$,

Case (1) $|\gamma_\nu| < 1$ $(\forall \nu \geq 0)$, *or*

Case (2) $\exists n$ *such that*

$$|\gamma_0| < 1, \quad |\gamma_1| < 1, \quad \dots, \quad |\gamma_{n-1}| < 1, \quad |\gamma_n| = 1.$$

In this case, $f_n(x) = c_{n0} + c_{n1} x + c_{n2} x^2 + \cdots$ *is reduced to the constant* $c_{n0} = \gamma_n$.

In Case (1), (f 無限階*)*

$$f(x) = [x; \gamma_0, \gamma_1, \gamma_2, \dots] = \sum_{\nu=0}^{\infty} \Psi(\gamma_0, \gamma_1, \dots, \gamma_\nu)\, x^\nu .$$

Case (2) occurs if and only if $f(x)$ *is a rational function of the form (II.5), and*

$$f(x) = [x; \gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n] \quad (\text{有限階}).$$

Satz III. **(Schur problem)** *Let* $c_0, c_1, \dots, c_m$ *given. Then a series*

$$f(x) = (c_0 + c_1 x + \cdots + c_m x^m) + (c_{m+1} x^{m+1} + c_{m+2} x^{m+2} + \cdots)$$

converging for $|x| < 1$, *with* $M(f) \leq 1$, *is given if and only if, for*

$$\gamma_\mu = \Phi(c_0, c_1, \dots, c_\mu) \quad (\mu = 0, 1, \dots, m),$$

Case (a) $|\gamma_\mu| < 1 \quad (0 \leq \mu \leq m)$, *or*

Case (b) $\quad \exists n \le m$ *such that*

$$|\gamma_0| < 1, \quad |\gamma_1| < 1, \quad \ldots, \quad |\gamma_{n-1}| < 1, \quad |\gamma_n| = 1,$$

and c_μ, $\mu = n+1, \ldots, m$ *coincide with coefficients in* $[x; \gamma_0, \gamma_1, \ldots, \gamma_n]$.

In Case (a), f is given by $f(x) = [x; \gamma_0, \gamma_1, \ldots, \gamma_m, \gamma_{m+1}, \ldots]$, with γ_μ, $|\gamma_\mu| \le 1$ $(\forall \mu \ge m+1)$. In Case (b), $f(x) = [x; \gamma_0, \gamma_1, \ldots, \gamma_n]$ is the unique solution.

Satz IV.

$$\mathfrak{E} = \Big\{ f(x) = [x; \gamma_0, \gamma_1, \gamma_2, \ldots] = \sum_{\nu=0}^{\infty} \Psi(\gamma_0, \gamma_1, \ldots, \gamma_\nu)\, x^\nu \ ; \ |\gamma_\mu| \le 1 \ \ (\forall \mu \ge 0) \Big\}.$$

In case $|\gamma_\mu| < 1$ $(\forall \mu \ge 0)$, *the parameter is unique. But in case of (II.5),* $\gamma_{n+1}, \gamma_{n+2}, \ldots,$ *are arbitrary.*

4. Calculation of the expression Φ_ν

$$f(x) = \frac{g(x)}{h(x)} = c_0 + c_1 x + c_2 x^2 + \cdots,$$

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \quad g_\nu(x) = a_\nu + a_{\nu+1} x + a_{\nu+2} x^2 + \cdots \quad (\nu \ge 0),$$

$$h(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \quad h_\nu(x) = b_\nu + b_{\nu+1} x + b_{\nu+2} x^2 + \cdots \quad (\nu \ge 0),$$

$$f_{\nu+1} = \frac{1}{x} \frac{f_\nu - \gamma_\nu}{1 - \overline{\gamma_\nu} f_\nu}, \quad f_\nu = \frac{x f_{\nu+1} + \gamma_\nu}{1 + \overline{\gamma_\nu}\, x f_{\nu+1}}, \quad \gamma_\nu = f_\nu(0).$$

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Satz V. $f_\nu(x)$ and $\gamma_\nu = f_\nu(0)$ are expressed as follows:

$$D_\nu(x) := \begin{vmatrix} 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{\nu-1} & g_\nu(x) \\ \overline{b_0} & 0 & \cdots & 0 & 0 & a_0 & \cdots & a_{\nu-2} & g_{\nu-1}(x) \\ \overline{b_1} & \overline{b_0} & \cdots & 0 & 0 & 0 & \cdots & a_{\nu-3} & g_{\nu-2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{b_{\nu-2}} & \overline{b_{\nu-3}} & \cdots & \overline{b_0} & 0 & 0 & \cdots & a_0 & g_1(x) \\ 0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{\nu-1} & h_\nu(x) \\ \overline{a_0} & 0 & \cdots & 0 & 0 & b_0 & \cdots & b_{\nu-2} & h_{\nu-1}(x) \\ \overline{a_1} & \overline{a_0} & \cdots & 0 & 0 & 0 & \cdots & b_{\nu-3} & h_{\nu-2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{a_{\nu-2}} & \overline{a_{\nu-3}} & \cdots & \overline{a_0} & 0 & 0 & \cdots & b_0 & h_1(x) \end{vmatrix} \quad \begin{matrix} (\nu+\nu) \\ \times[(\nu-1)+(\nu+1)] \end{matrix}$$

$$\Delta_\nu(x) := \begin{vmatrix} \overline{b_0} & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{\nu-1} & g_{\nu-1}(x) \\ \overline{b_1} & \overline{b_0} & \cdots & 0 & 0 & a_0 & \cdots & a_{\nu-2} & g_{\nu-2}(x) \\ \overline{b_2} & \overline{b_1} & \cdots & 0 & 0 & 0 & \cdots & a_{\nu-3} & g_{\nu-3}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{b_{\nu-1}} & \overline{b_{\nu-2}} & \cdots & \overline{b_0} & 0 & 0 & \cdots & 0 & g_0(x) \\ \overline{a_0} & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{\nu-2} & h_{\nu-1}(x) \\ \overline{a_1} & \overline{a_0} & \cdots & 0 & 0 & b_0 & \cdots & b_{\nu-3} & h_{\nu-2}(x) \\ \overline{a_2} & \overline{a_1} & \cdots & 0 & 0 & 0 & \cdots & b_{\nu-4} & h_{\nu-3}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{a_{\nu-1}} & \overline{a_{\nu-2}} & \cdots & \overline{a_0} & 0 & 0 & \cdots & 0 & h_0(x) \end{vmatrix} \quad (\nu+\nu)\times(\nu+\nu)$$

Put $d_\nu := D_\nu(0)$, $\delta_\nu := \Delta_\nu(0)$. If $\delta_\lambda \neq 0$ $(\lambda = 1, 2, \ldots, \nu-1)$, then

$$f_\nu(x) = \frac{1}{x}\,\frac{f_{\nu-1} - \gamma_{\nu-1}}{1 - \overline{\gamma_{\nu-1}} f_{\nu-1}} = -\frac{D_\nu(x)}{\Delta_\nu(x)}, \quad \gamma_{\nu-1} = f_{\nu-1}(0) = -\frac{d_{\nu-1}}{\delta_{\nu-1}}.$$

余因子行列に関する *Jacobi*'s Theorem を用いると，次が分かる：

$$\text{(II.6)} \qquad d_\nu \Delta_\nu(x) - \delta_\nu D_\nu(x) = -\delta_{\nu-1} x\, D_{\nu+1}(x),$$

(II.7) $$\delta_\nu \Delta_\nu(x) - \overline{d_\nu} D_\nu(x) = \delta_{\nu-1}\Delta_{\nu+1}(x),$$

(II.7) 式は，$x = 0$ において次の重要な式を与える：

(II.8) $$1 - |\gamma_\nu|^2 = \frac{\delta_{\nu-1}\delta_{\nu+1}}{\delta_\nu^2} \qquad \Big(\delta_0 = 1,\ \delta_{-1} = \frac{1}{b_0^2}\Big).$$

5. Hermitian form belonging to the quotient of two power series

行列式 δ_ν の簡単な表示を求める.

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \overline{A}' = \begin{pmatrix} \overline{a_0} & 0 & 0 & \cdots \\ \overline{a_1} & \overline{a_0} & 0 & \cdots \\ \overline{a_2} & \overline{a_1} & \overline{a_0} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \infty \times \infty,$$

Cut off of A and $\overline{A}'$:

$$A_\nu := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \overline{A}'_\nu := \begin{pmatrix} \overline{a_0} & 0 & 0 & \cdots \\ \overline{a_1} & \overline{a_0} & 0 & \cdots \\ \overline{a_2} & \overline{a_1} & \overline{a_0} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (\nu+1) \times (\nu+1),$$

Hermite 行列 $\overline{A}'A$, そのcutoff $\overline{A}'_\nu A_\nu$,

Hermite 形式 $$\mathfrak{A}_\nu := \mathfrak{A}(x_0, x_1, \ldots, x_\nu) = \sum_{\lambda=0}^{\nu} \big|a_0 x_\lambda + a_1 x_{\lambda+1} + \cdots + a_{\nu-\lambda} x_\nu\big|^2$$

$$AB \longleftrightarrow g(x)h(x) = (a_0x_0 + a_1x_1 + \cdots)(b_0x_0 + b_1x_1 + \cdots)$$

Theorem. $\delta_{\nu+1} = \begin{vmatrix} \overline{B}'_\nu & A_\nu \\ \overline{A}'_\nu & B_\nu \end{vmatrix}$ *is equal to* $|\overline{B}'_\nu B_\nu - \overline{A}'_\nu A_\nu|$, *which corresponds to*

$$\mathfrak{H}_\nu = \mathfrak{H}(x_0, x_1, \ldots, x_\nu) = \mathfrak{B}_\nu - \mathfrak{A}_\nu =$$

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$$= \sum_{\lambda=0}^{\nu} \left(\left| b_0 x_\lambda + b_1 x_{\lambda+1} + \cdots + b_{\nu-\lambda} x_\nu \right|^2 - \left| a_0 x_\lambda + a_1 x_{\lambda+1} + \cdots + a_{\nu-\lambda} x_\nu \right|^2 \right).$$

$$f = \frac{g(x)}{h(x)} \rightsquigarrow \mathfrak{H}_\nu \ (\nu = 0, 1, 2, \ldots) \quad \text{(Hermitian forms)}.$$

Satz VI. *Let* $\delta_1 \neq 0,\ \delta_2 \neq 0,\ \ldots,\ \delta_n \neq 0$. *Then,*

$$\delta_\lambda = 0 \ (\forall \lambda > n) \iff f_n(x) \equiv \varepsilon \ (|\varepsilon| = 1).$$

Satz VII. *Hermitian form* $\mathfrak{H}$ *is non-negative if and only if*

Case (1) $\delta_\nu > 0 \ (\forall \nu \geq 1)$,

or

Case (2) $\delta_\nu > 0 \ (1 \geq n)$, $\delta_{n+\lambda} = 0 \ (\lambda \geq 1)$. *In this case,* $n = \operatorname{rank} \mathfrak{H}$.

◆一息入れてまとめ◆

$$1 - |\gamma_\nu|^2 = \frac{\delta_{\nu-1}\delta_{\nu+1}}{\delta_\nu^2} \qquad \left(\delta_0 = 1,\ \delta_{-1} = \frac{1}{b_0^2}\right),$$

Schur's algorithm:

$$\begin{aligned} f(x) &= c_0 + c_1 x + c_2 x^2 + \ldots \in \mathfrak{C}, \\ f_0(x) &:= f(x), \\ f_{\nu+1}(x) &= \frac{1}{x} \frac{f_\nu(x) - \gamma_\nu}{1 - \overline{\gamma_\nu} f_\nu(x)}, \qquad \gamma_\nu = f_\nu(0), \\ f_\nu(x) &= \frac{x f_{\nu+1}(x) + \gamma_\nu}{1 + \overline{\gamma_\nu}\, x f_{\nu+1}(x)}. \end{aligned}$$

Schur parameters:

$\gamma_\nu := c_{\nu 0} := \Phi(c_0, c_1, \ldots, c_\nu) =: \Phi_\nu$,

a rational function of $c_0, \overline{c_0}, c_1, \overline{c_1}, \ldots c_{\nu-1}, \overline{c_{\nu-1}}, c_\nu$.

$c_\nu = \Psi(\gamma_0, \gamma_1, \ldots, \gamma_\nu) =: \Psi_\nu$

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a polynomial function of $\gamma_0, \overline{\gamma_0}, \gamma_1, \overline{\gamma_1}, \ldots \gamma_{\nu-1}, \overline{\gamma_{\nu-1}}, \gamma_\nu$,

Schur's matrix product:

$$A = (a_{\kappa\lambda}),\ B = (b_{\kappa\lambda}), \quad A * B := \big(a_{\kappa\lambda} b_{\kappa\lambda}\big).$$

6. Deformation of the criterion in §3

$$f(x) = \frac{g(x)}{h(x)} \implies \gamma_\nu = -\frac{d_\nu}{\delta_\nu},$$

(II.9) $$|\gamma_0| < 1,\ |\gamma_1| < 1, \ldots,\ |\gamma_{n-1}| < 1,\ \gamma_n| \leq 1,$$

iff, from(3),

$$\delta_1 > 0,\ \delta_2 > 0,\ \ldots,\ \delta_n > 0,\ \delta_{n+1} \geq 0.$$

If $\delta_{n+1} = 0$, then $|\gamma_n| = 1$ and

$$f_n(x) = -\frac{D_n(x)}{\Delta_n(x)} = \gamma_n = c_{n0} + c_{n1}x + \cdots,$$

and $\delta_\lambda = 0\ (\forall\lambda > n)$ by Satz VI.

Satz VIII (Satz II の別表現). $f(x) = \dfrac{g(x)}{h(x)}$ の冪級数展開において，それが，$|x| < 1$ で収束し，$M(f) \leq 1$ となるための必要十分条件は，

$$\delta_1 = \begin{vmatrix} \overline{b_0} & a_0 \\ \overline{a_0} & b_0 \end{vmatrix}, \quad \delta_{\nu+1} = \begin{vmatrix} \overline{B}'_\nu & A_\nu \\ \overline{A}'_\nu & B_\nu \end{vmatrix} \quad (\nu \geq 1),$$

(1) $\delta_\nu > 0 \quad (\forall\nu \geq 0)$,

or

(2) $\delta_1 > 0,\ \ldots,\ \delta_n > 0,\ \delta_{n+1} = \delta_{n+2} = \cdots = 0.$

In this case,

(II.10) $$f(x) = \varepsilon \prod_{\nu=1}^{n} \frac{x + \omega}{1 + \overline{\omega}\, x}, \quad |\omega_\nu| < 1,\ |\varepsilon| = 1.$$

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Satz VIII* (from Satz III). $f(x)=\dfrac{g(x)}{h(x)}$ の冪級数展開において，それが $|x|<1$ で収束し，$M(f)\leq 1$ となるための必要十分条件は，*Hermite* 形式 $\mathfrak{H}=\overline{B}'B-\overline{A}'A$ が非負形式であること．

$f(x)$ が有限階数 n であるための必要十分条件は，*Hermite* 形式 $\mathfrak{H}$ が階数 n であること．このとき，$f(x)$ は，*(II.10)* の形である．

Satz IX (Extension of Satz III).

$$G(x)=\sum_{\nu=0}^{\infty}k_\nu x^\nu,\quad H(x)=\sum_{\nu=0}^{\infty}l_\nu x^\nu,\quad l_0\neq 0,$$

$$g(x)=\sum_{\nu=0}^{\infty}a_\nu x^\nu,\quad h(x)=\sum_{\nu=0}^{\infty}b_\nu x^\nu,\quad b_0\neq 0,$$

(II.11) *for* $m\geq 0$ *given,* $a_i=k_i,\ b_i=l_i\ (0\leq i\leq m),\quad f(x)=\dfrac{g(x)}{h(x)},$

$f(x)$ の冪級数展開が $|x|<1$ で収束し，$M(f)\leq 1$, となるための十分条件を与えるのに，

$$F(x)=\frac{G(x)}{H(x)}=C_0+C_1x+C_2x^2+\cdots,$$

を使う．$f(x)$ に対する $\delta_1,\delta_2,\dots,$ と同様に，

$$\eta_1=\begin{vmatrix}\overline{b_0} & a_0\\ \overline{a_0} & b_0\end{vmatrix},\quad \eta_2,\ \eta_3,\ \dots$$

を作る．十分条件は，２つの場合に分かれる：

(1) $\eta_\nu>0\quad(\forall\nu\geq 0)$,

or

(2) $\eta_1>1,\ \dots,\ \eta_n>1$, *and*

(II.12) $$\eta_{n+1}=\eta_{n+2}=\cdots=\eta_{m+1}=0.$$

Here if $n<m-1$, *then we have in addition,*

(II.13) $$\eta_{m+2}=\eta_{m+3}=\cdots=\eta_{2m-n}=0.$$

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In both cases, $b_{m+1}, b_{m+2}, \dots$ *can be arbitrary.*

In Case (1), solutions $f(x)$ *are infinitely many.*

In Case (2), solution $f(x)$ *is unique and given by (II.10).* $a_{m+1}, a_{m+2}, \dots$ *are unique for a fixed series of* $b_{m+1}, b_{m+2}, \dots$.

7. Bounded power series and bounded bilinear forms

From Satz VIII and Satz VIII*,

Satz X. *The power series* $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ *converges on* $|x| < 1$ *and bounded if and only if*

$$A(x,y) = \sum_{\kappa \leqslant \lambda} a_{\lambda-\kappa} x_\kappa y_\lambda, \quad A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

is bounded, and $M(g) = \|A\| := \sup_{\|x\|, \|y\| \leqslant 1} \big|A(x,y)\big|$.

Satz X*. *Put* $H := A^* A$. *Then* $m \geq M(g)$ *is characterized by*

$$\mathfrak{H} := m^2 - A^* A \geq 0. \quad \big(\text{here } \mathfrak{H}_\nu = (m^2 - A^* A)_\nu = m^2 E_{\nu+1} - A_\nu^* A_\nu\big)$$

This is the case if and only if, for the sectional determinants $\delta_1, \delta_2, \dots$ *of* $\mathfrak{H}$

(1) $\delta_\nu > 0 \ (\forall \lambda \geq 1)$,

or

(2) $\delta_1 > 0,\ \delta_2 > 0,\ \dots,\ \delta_n > 0,\ \delta_{n+1} = 0.$ *In this case,* $n = \operatorname{rank} \mathfrak{H}$, *and*

$$g(x) = c \prod_{\nu=1}^{n} \frac{x + \omega_\nu}{1 + \overline{\omega_\nu}\, x}, \quad |\omega_\nu| < 1,\ |c| = 1.$$

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8. *Carathéodory-Toeplitz*'s Theorem

$$w' = \frac{1-w}{1+w}: \quad \text{半平面}\ \Re(w) > 0 \ \longrightarrow\ \text{単位円盤}\ |w'| < 1; \quad w = \frac{1-w'}{1+w'},$$

$$f(x) = \frac{1-\varphi(x)}{1+\varphi(x)}, \quad \varphi(x)\ \text{analytic on}\ |x|<1\ \text{and}\ \Re\varphi(x) > 0$$

$$\Longleftrightarrow\ f(x)\ \text{analytic on}\ |x|<1\ \&\ |f(x)|<1.$$

$$\varphi(x) = g(x)/h(x), \quad g(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu, \quad h(x) = \sum_{\nu=0}^{\infty} b_\nu x^\nu, \quad \text{とおくと,}$$

$$f(x) = \frac{h(x)-g(x)}{h(x)+g(x)}, \quad \text{corresponding matrices} \quad B-A\&B+A,$$

Hermitian form: $\mathfrak{H} = (B+A)^*(B+A) - (B-A)^*(B-A) = 2(B*A + A^*B)$.

Put $h(x)=1,\ g(x)=\varphi(x)$,

$$\frac{1}{2}\mathfrak{H} = A + A^* = \begin{pmatrix} 2a_0' & a_1 & a_2 & a_3 & \dots \\ \overline{a_1} & 2a_0' & a_1 & a_2 & \dots \\ \overline{a_2} & \overline{a_1} & 2a_0' & a_1 & \dots \\ \overline{a_3} & \overline{a_2} & \overline{a_1} & 2a_0' & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \left(\mu_{j-k}\right)_{j,k=0}^{\infty},$$

$$a_0 + \overline{a_0} = 2a_0',\ a_0' := \Re a_0,\ \mu_0 = a_0 + \overline{a_0},\ \mu_j = a_j,\ \mu_{-j} = \overline{\mu_j}\ (j \geq 1),$$

Satz XI (from Satzes VIII, VIII*). *Power series* $\varphi(x) = a_0 + a_1x + a_2x^2 + \cdots$ *is*

convergent for $|x|<1$ *and* $\Re\varphi(x) > 0$

$\Longleftrightarrow$

for sectional determinants $\delta_1 = 2a_0'$, $\delta_2 = \begin{vmatrix} 2a_0' & a_1 \\ \overline{a_1} & 2a_0' \end{vmatrix}, \dots, \delta_\nu = \det(\mathfrak{H}_\nu), \dots,$

(1) $\delta_\nu > 0\ (\forall \nu \geq 1)$,

or

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(2) $\delta_\nu > 0\ (\forall \nu \le n),\ \delta_\nu = 0\ (\forall \nu > n)$. *In this case,*

$$\text{(II.14)} \qquad \varphi(x) = \frac{1-f(x)}{1+f(x)}, \quad f(x) = \varepsilon \prod_{\nu=1}^{n} \frac{x+\omega_\nu}{1+\overline{\omega_\nu}\, x} \quad (|\omega_\nu| < 1,\ |\varepsilon| = 1).$$

By Carathéodory

$$\text{(II.15)} \qquad \varphi(x) = bi - \sum_{\nu=1}^{n} r_\nu \frac{x+\varepsilon_\nu}{x-\varepsilon_\nu},$$

$$(b \in \boldsymbol{R},\ r_\nu > 0,\ \varepsilon_\nu \text{ all different}, |\varepsilon_\nu| = 1).$$

• (II.14) $\Longrightarrow$ (II.15) の証明.

$$Q(x) := a \prod_{\nu=1}^{n} (x+\omega_\nu), \quad |\omega_\nu| < 1),$$

$$P(x) := x^n Q(x^{-1}) = \overline{a} \prod_{\nu=1}^{n} (1+\overline{\omega_\nu}\, x),$$

$$f(x) = \frac{Q(x)}{P(x)},$$

$$\varphi(x) = \frac{1-f(x)}{1+f(x)} = \frac{P-Q}{P+Q} \qquad \square$$

Satz XII. *The n roots of*

$$\text{(II.16)} \qquad P(x) + Q(x) = 0,$$

are all different each other and on the unit circle. Partial fractional expansion is given as

$$\text{(II.17)} \qquad \frac{P-Q}{P+Q} = c - \sum_{\nu=1}^{n} r_\nu \frac{x+\varepsilon_\nu}{x-\varepsilon_\nu},$$

$$(c \in i\boldsymbol{R},\ r_\nu > 0,\ \varepsilon_\nu \text{ all different}, |\varepsilon_\nu| = 1).$$

• Satz XI $\Longrightarrow$ Satz VIII, VIII*, 故に, Satz XI $\Longleftrightarrow$ Satz VIII, VIII*

(論文 [S29] 終了)

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[S30] J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. II

9. An application of Satz IV

Landau [Land1913] の定理の一般化.

Satz XIII. *Let* $S(x_0, x_1, \ldots, x_n)$ *be a given polynomial* $\neq$ *constant, and*

$$G := \sup_{f\in\mathfrak{E}} |S(c_0, c_1, \ldots, c_n)|, \qquad f(x) = c_0 + c_1 x + c_2 x^2 + \cdots .$$

Then $\exists f \in \mathfrak{E}$ *such that* $|S(c_0, c_1, \ldots, c_n)| = G$ *and any such* f *is of the form*

$$f(x) = \varepsilon \prod_{\nu=1}^{r} \frac{x + \omega_\nu}{1 + \overline{\omega_\nu}\, x} \quad (|\omega_\nu| < 1, \ \ |\varepsilon| = 1),$$

where $r \leq n$ *is possible.*

10. A consequence of Satz X and Satz XI

Theorem Schur 1 [Sch1911, p.11]. *Let*

$$A = \sum_{\kappa,\lambda=1}^{\infty} a_{\kappa\lambda}\, x_\kappa\, x_\lambda, \quad B = \sum_{\kappa,\lambda=1}^{\infty} b_{\kappa\lambda}\, x_\kappa\, x_\lambda,$$

be non-negative definite Hermitian form. Then

$$\text{(Schur product)} \qquad C = \sum_{\kappa,\lambda=1}^{\infty} a_{\kappa\lambda} b_{\kappa\lambda}\, x_\kappa\, x_\lambda,$$

is also non-negative definite.

Theorem Schur 2 [Sch1911, p.14]. *Let*

$$A = \sum_{\kappa,\lambda=1}^{\infty} a_{\kappa\lambda}\, x_\kappa\, x_\lambda, \quad B = \sum_{\kappa,\lambda=1}^{\infty} b_{\kappa\lambda}\, x_\kappa\, x_\lambda,$$

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be resp. bounded, and a non-negative definite Hermitian form. If $b := \sup_{i \geqslant 1} b_{ii} < \infty$, *then*

$$C = \sum_{\kappa,\lambda=1}^{\infty} a_{\kappa\lambda} b_{\kappa\lambda}\, x_\kappa\, x_\lambda,$$

is bounded and $\|C\| \leq b\,\|A\|$.

Satx XIV. *Let*

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu, \quad g(x) = \sum_{\nu=0}^{\infty} b_\nu x^\nu,$$

which converge on $|x| < 1$, *and* $\Re\big(g(x)\big) > 0$, *and put*

$$h(x) := 2a_0 b_0' + \sum_{\nu=0}^{\infty} a_\nu b_\nu x^\nu, \quad b_0' := \Re b_0,$$

converging in $|x| < 1$. *It has* $\Re\big(h(x)\big) > 0$ *(resp. bounded) when* $f(x)$ *has the same property. In the second case,*

$$M(h) \leq 2b_0'\, M(f). \tag{II.18}$$

Proof. By Theorems Schur 1 and 2, and Satz X and Satz XI. □

(§10 終了，以下省略)

11. On partial sums of bounded power series

12. On a special class of bounded power series

13. On polynomials, which take zeros only inside of unit disk

14. The rational function $[x; \gamma_0, \gamma_1, \ldots, \gamma_n]$

15. Some properties of the parameter expression of for a bounded power series

References

[注 意] J. Schur の論文 [S29], [S30] の文中もしくは脚注に引用されている文献をまとめた．彼の論文としては引用が異例の多さである．主題が主題だからかも知れない．各引用では必要最小限の書誌情報しか与えられていないので，欠けている論文タイトル等を補って出来るだけ完全な情報にしようとしたが，時間の関係で未完である．なお，各論文の書誌情報のうしろで独立した括弧内には，例えば (p.417, §13) とあるが，p.417 は被引用論文の指定頁，§13 は引用した節，を表す．また，見出しの * 印（例えば [*Biehler]）は引用が論文 [S30] で起こっていることを示す．

[*Biehler] C. Biehler, Sur une classe d'équations algébriques dont toutes les racines sont réelles, Journ. für die reine und angewante Mathmathematik, **87** (1879), 350–352. (§13)

[*Blas1915] W. Blaschke,???, Leipziger Berichte, **LXVII** (1915), ???– (p.194, §12)

[*Bohr1917] H. Bohr, ???, Gött. Nachr., math.-phys.Klas. **1917**, pp.119–128. (§11)

[Cara1907] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten vom Potenzreihen, die gegebene Werte nicht annehmen, Math. Ann., **64** (1907), 95–115. (§0)

[Cara1911] C. Carathéodory, Über die Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen, Rend. di Palermo, **32** (1911), 193–217. (§0, §8)

[CaFe1911] C. Carathéodory and L. Fejér, Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landau'schen Satz, Rend. di Palermo, **32** (1911), 218–239. (§0, §3)

[*delaV1908] de la Vallée-Poussin, ???, Bull. de l'Acad. Sc.de Belgique **1908**, ???– (p.193, §11)

[*Fatou1906] P. Fatou, ???, Acta Math., **30** (1906), ???– (p.335, §12)

[*Fej1908] L. Fejér, Über die Wurzel vom kleinsten absoluten Betrage einer algebraischen Gleichung, Math. Ann., **65** (1908), 413–423. (p.417, §13)

[*Fej1910] L. Fejér, ???, Münchener Ber., **1910**, Nr. 3, (§10)

[*Fej1915] L. Fejér, Über trigonometrische Polynome, J. f. d. reine u. angew. Math., **146** (1915), 53–82 (p. 59, §11)

57

[*Fej1914] L. Fejér, Über gewisse durch die Fouriersche und Laplacesche definierten Mittelkurven und Mittelflächen, Rendiconti di Palermo, **38** (1914), 79–97. (p. 95, §11)

[Fish1911] E. Fisher, Über das Carathéodory'sche Problem, Potenzreihen mit positiven reellen Teil betreffend, Rend. di Palermo, **32** (1911), 240–256. (§0)

[Frob1912] G. Frobenius, Ableitung eines Satzes von Carathéodory aus einer Formel von Kronecker, Berl. Berichte, **1912**, 16–31. (§0)

[*Gauss1866] J. Gauß, Werke, **III**, **1866**. (p.112, §13)

[*Gron1912] T.H. Gronwall, ???, Annals of Math., **14** (1912), ???– (§1, §12)

[Gron1914] T.H. Gronwall, ???, Annals of Math., **16** (1914), ???– (p.77, §0)

[*Gron1916] T.H. Gronwall, Über einige Sumationsmethoden und ihre Anwendung auf die *Fourier*sche Reihe, J. f.d. r. u. angew. Math., **147** (1916), p.16–35. (§11)

[Herg1911] G. Herglots, ???, Leipz. Berichte, **1911**, ???– (p.501, §0)

[HelTo1911] E. Hellinger und O. Toeplitz, Grundlagen für eine Theorie der unendlichen Matrizen, Math. Ann., **69** (1910), 289-330 (§7, §9).

[Hilb1906] D. Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Göttingen Nachrichten, **1906**, pp.157-227. (§7)

[*Hurw1895] A. Hürwitz, Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt, Math. Ann., **46** (1895), 273-284 (§13)

[Land1906] E. Landau, ???, Vierteljarsschrift der Züricher Naturf. Ges. **51** (1906), ???– (p.252, §1)

[*Land1913] E. Landau, ???, Archiv d. Math. u. Phys. **21** (1913), ???– (§9, §11)

[*Land???] E. Landau, Handbuch der Lehre von der Verteilung der Primzahren, ???– (p.891, §10)

[*Land1916] E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, **1916**, ???– (p.20, §9)

[*Orla1912] L. Orland, Sal problema di Hurwitz relativo alle parti reali dello radici de un' equazione algebrica, Math. Ann., **71** (1912), 233-245. (§13)

58

[Pick1915] G. Pick, Über die Beschrä nkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden, Math. Ann., **77** (1915), 7–23. (§0, §2)

[Ries1911] F. Riesz, ???, Ann. de l'École Norm., **1911**, ???– (p.33, §0)

[*Ries1915] F. Riesz, Über ein Problem des Herrn *Carathéodory*, Journ. f. d. r. u. angew. Math., **146** (1915), 83–87. (§0, §10)

[*Routh1898] E.J. Routh, *Die Dynamik der Systeme starrer Körper*, 独語版, Leipzig, 1898, Bd. 2, §291. (§13)

[Sch1911] [S19] J. Schur, Bemerkungen zur Thorie der beschränkten Bilinearformen mit unentlich vielen Veränderlichen, Journ. f. d. r. u. angew. Math., **140** (1911), 1–28. (§10)

[Sch1912] [S21] I. Schur, Über einen Satz von C. Carathéodory, Berliner Berichte **1912**, ??– (p.4, §0)

[*Schur1917] [S29] J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. I, *J. für die reine und angewandte Mathematik*, **147**(1917), 205–232 (§§1–8).

[*Study1913] E. Study, *Konforme Abbildung einfach-zusammenhängender Bereiche*, Teubner, 1913, ???– (p.50, §12)

[Toep1910] O. Toeplitz, ???, Gött. Nachrichten, **1910**, ???– (p.489, §0)

[Toep1911] O. Toeplitz, Über die Fourier'sche Entwickelung positiver Funktionen. (Aus einem an Herrn Carathéodory gerichteten Briefe), Rend. di Palermo, **32** (1911), pp.191–192. (§0)

[Toep1911] O. Toeplitz, Zur Theorie der quadratischen und bilinearen Formem von unendlichvielen Veränderlichen, I. Teil: Theorie der L-Formen, Math. Ann., **70** (1911), 351–376. (§5, §8)

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