# *FLUID DYNAMICS AND HEAT THEORY BY POISSON*
# *POISSON* の流体力学と熱理論

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Abstract.

We discuss the fluid dynamics and the heat theory by Poisson. These two theories or themes are studied in the arrival of continuum.

The hydrodynamists like Navier, Poisson and Cauchy propose at first the wave equations in the elasticity and next, the fluid equations in incompressibility since Euler and Lagrange succeeded. Poisson and Navier discuss the activity of molecules in regard to the attraction and repulsion in rivalry to each other. Navier depends on the Fourier's idea comes from the theory on heat analysis. Fourier and Poisson propose the heat theories in rivalry to each other, from the viewpoint of mathematical physics on the continuum. These all are, the mathimatical physicians, but we think, Poisson is an acuter and severer observer on the physics than others. From here, the both theories and deductive method of heat theory are different, however, the results are the same one. Although regretably, Poisson misses the historical priority in the fluid dynamics and the heat theory, however, contributes to reform the preceding theories.

Our motivation in this paper is to consider the Poisson's singularity from the mathematical viewpoints.

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Key words. The Navier-Stokes equations, fluid dynamics, fluid mechanics, hydrostatics, hydrodynamics, hydromechanics, thermodynamics, heat difusion equations, wave equations, Kepler problems, Schrödinger equations, microscopically-descriptive equations, mathematical history, physico-mathematics, trigonometric series, Fourier series, quantum mechanics.

## Contents



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## 1. Introdiction

[1,2,3]

### 1.1. What is the wave, the heat and the fluid ?

Kepler (1571-1630) 1634 [34] proposes the laws on motions of planets in reserving many analytical open problems. Huygens (1629-95) proposes and Fresnel (1788-1827) corrects the wave principles. Euler 1748 [16] proposes the wave motion of string. Navier [52] and Poisson [73] propose the fluid equations, successively after the erastic wave equations of Navier's [51] and Poisson's [73] respectively. After Fourier 1822 [22] completes the heat theory, Fourier 1833 [28] combines his communication theory with the Euler equation 1755 [17] and puts the heat equation of motion in fluid, in which he expresses the molecular motion with communication and transportation of molecules before Boltzmann's modeling with collision and transportation.

How does the wave occur ? Newton 1686 [55] shows his principle on the wave motion in the water pressure.

> The pressure doesn't propagate by the fluid of the secondary linear strait, except for the particle of adjacent fluid. If the adjacent particles $a, b, c, d, e$

[1] Translation from Latin/French/German into English mine, except for Boltzmann.

[2] To establish a time line of these contributor, we list for easy reference the year of their birth and death: Newton (1643-1727), Euler (1707-83), d'Alembert (1717-83), Lagrange (1736-1813), Laplace (1749-1827), Fourier (1768-1830), Navier (1785-1836), Poisson (1781-1840), Cauchy (1789-1857), Dirichlet (1805-59), Riemann (1826-66), Boltzmann (1844-1906), Hilbert (1862-1943), Schrödinger (1887-1961).

[3] We use ($\Downarrow$) means our remark not original, when we want to avoid the confusions between our opinion and sic. ($\Leftarrow$) means our translation in citing the origin.

propagate in the straight line, press from $a$ to $e$ ; the particle $e$ progresses separately into the oblique points $f$ and $g$, and without sustained pressure, and moreover, to the particles $h$ and $k$ ; m as it is fixed in another direction, it presses for the particle into propping up ; the unsustained pressure goes separately into the particles $l$ and $m$, and as this way, it follows successively and limitlessly. thus it will occur so many time, inaccurately, to the particle in the indirect adjacency. Q.E.D. [55, pp.354-5] (trans. from Latin, mine.)

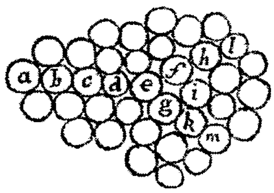

**fig.1 The wave pushed with particle by Newton's hypothesis in 1686**

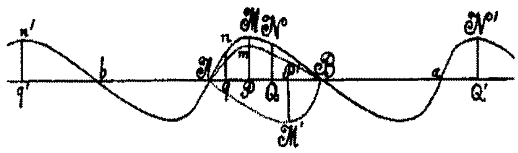

**fig.2 The wave of a cord by Euler in 1748**

What is the fluid ? According to today's diffinition, it is called the fluid is a *limitlessly free continuum.* Where does *continuum* come from in the historical view ?

### 1.2. General remark. Fluid Dynamics and Heat Theory by Poisson.

1. We discuss historical development of classical fluid dynamics and heat theory from the viewpoint of mathematical history, in particular, of Poisson.
2. These situations owe to the arrival of *continuum*, on which we summarize the trail-blazers of the trigonometric series such as Euler, Lagrange, Laplace, et al.
3. Poisson issues his last work [77] in 1835 in rivalry to Fourier and Navier, in which he discusses the essential theories for the expression between fluid motion and heat motion, emphasizing the hypothesis of molecular radiation with the mathematical points such as complete integral.
4. Prévost's work [81] on heat communication, which precedes Fourier, and whose initial scholar work and after it.
5. Sturm and Liouville refer Poisson's tools such as particular value and particular function, entire function, to solve the differential problems.
6. Comparing these books and papers, we show the connection between the hydrodynamics, wave and heat dynamics, and the process of new mathematics putting forth in applied or physical mathematics.

### 1.3. Poisson's paradigm and singularity.

Poisson publishes the last books consist of three elements : [74, 75, 76, 77]. ([75, 76] are the same title and are divided into two volumes.) These are his paradigm of the mathematical physics through all his academic life, entitled a study of mathematical phisics. (*Un Traité de Physique Mathèmatique.*) In the rivalry to Euler, Lagrange, Laplace, Fourier, Navier, et al., we think, he struggles to make his paradigm. On the other hand, as its proofs, there are some singular but important sugestions such as :

- rigorous sum instead of integral,
- critics to easy applying the rule comes from real to transcendental function,
- conjecture on the defect of the proof in the eternity of exact differential,
- contribution to the fluid dynamics, especially, to the Navier-Stokes equations,
- deduction of another heat equation from the basically molecular analysis.

We discuss these topics in the following papers.

## 2. The heat and fluid theories in the 19th century

### 2.1. The theory of heat communication in the Prévost's essay.

Prévost [81] discuss the communication of heat between two corps in earlier than Fourier, who corresponds with Prévost, according to Grattan-Guiness [33, p.23].

His principles are as follows : all the corps radiate the heat without relation to the temperature. The heat equilibrium is induced with the equal quantity of heat by the heat communication. These principles become shared with Fourier successively. (cf. Table 3.)

### 2.2. The outline of the situations surrounding Fourier and Poisson.

About the situations around Fourier, we can summarize as follows :

1. Fourier's manuscript 1807, which had been unknown for us until 1972, I. Grattan-Guinness [33] discovered it. Fourier's paper 1812 based on the manuscript was prized by the academy of France. We consider that Fourier, in his life work of the heat theory, begins with the communication theory, and he devoted in establishing this theme as the priority.
2. Owing to the arrival of continuum theory, many mathematical physical works are introduced, such as that Fourier and Poisson struggle to deduce the trigonometric series in the heat theory and heat diffusion equations. In the curent of formularizing process of the fluid dynamics, Navier, Poisson, Cauchy and Stokes struggle to deduce the wave equations and the Navier-Stokes equations. Of cource, there are many proceding researches before these topics, however, for lack of space, we must pick up at least, the essentials such as following contents :
3. Fourier [28] combines heat theory with the Euler's equations of incompressible fluid dynamics and proposes the equation of heat motion in fluid in 1820, however, this paper was published in 1833 after 13 years, it was after 3 years since Fourier passed away. Fourier seems to have been doutful to publish it in life.
4. After Fourier's commnunication theory, the gas theorists like Maxwell, Kirchhoff, Boltzmann [4] study the transport equations with the concept of collision and transport of the molecules in mass. In both principles, we see almost same relation between the Fourier's communication and transport of heat molecules and the Boltzmann's collision and transport of gas molecules.
5. Since 1811, Poisson issued many papers on the definite integral, containing transcendental, and remarked on the necessity of careful handling to the diversion from real to imaginary, especially, to Fourier explicitly. To Euler and Laplace, Poisson owes many knowledge, and builds up his principle of integral, consulting Lagrange, Lacroix, Legendre, etc. On the other hand, Poisson feels incompatibility with Laplace's 'passage', on which Laplace had issued a paper in 1809, entitled : On the 'reciprocal' passage of results between real and imaginary. in 1782-3.
6. To these passages, Poisson proposed the direct, double integral in 1811, 13, 15, 20 and 23. The one analytic method of Poisson 1811 is using the round braket, contrary to the Euler's integral 1781. The multiple integral itself was discussed and practical by Laplace in 1782, about 20 years before, when Poisson applied it to his analysis in 1806.
7. As a contemporary, Fourier is made a victim by Poisson. To Fourier's main work :

*The analytical theory of heat* in 1822, and to the relating papers, Poisson points the diversion applying the what-Poisson-called-it 'algebraic' theorem of De Gua or the method of cascades by Roll, to transcendental equation. Moreover, about their contrarieties, Darboux, the editor of *Œuvres de Fourier*, evaluates on the correctness of Poisson's reasonings in 1888. Drichlet also mentions about Fourier's method as a sort of *singularity of passage* from the finite to the infinite.

## 2.3. The preliminary discources on Fourier from the Nota to I.Grattan-Guinness.

### 2.3.1. *The Fourier's Oeuvres edited by G. Darboux.*

The preliminary discource by Fourier, edited by G. Barboux, says in 1820 :

> G. Darboux says in his first edition in 1888 : The works relating to the heat theory by Fourier appear in the late 18th century. It has been submitted to the Academy of Science, in Dec. 21, 1807. his first publication is unknown for us : we don't know except for an extract of 4 pages of BSP in 1808 ; It was read by the Committee, however, may be withdrawn by Fourier during 1810. The Committee of Academy, held in 1811, decided the following judgment : "Make clear the mathematical theory on the propagation of heat, and compare this theory with the exact result of experiments." (trans. mine.) [4]

### 2.3.2. *The Fourier 1822 by A. Freeman and The Fourier 1807 edited by I. Grattan-Guinness.*

In 1878, A. Freeman published the first English translated Fourier's second version, of which the preliminary is completely the same as G. Darboux 1888, ten years later than A. Freeman. In 1972, I. Grattan-Guinness discovered the manuscript 1807. He pays attentions to the Avertissment in the second edition by G. Darboux as above we mention. We are thankful to Grattan-Guinness for the showing one of the paragraph of ¶.136 ( Des températures finales et de la courbe qui les présente. ), and its belonging figure[5] of the Fourier's Manuscript 1807, *Théorie de la propagation de la chaleur*, edited and commented by Grattan-Guinness [33, p.371-2].

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[4](⇓) About the extract, same as above footnote. Lagrange was a member of the Committee of judgement and poses against Fourier's paper 1807. cf [83]. G.Darboux lists as follows : Lagrange, Laplace, Malus, Haüe and Legendre. [10, p.vii].

[5]This figure is the Fourier's original. [33, p.370]. In this figure, on the $x$ axis, there are the numbers 1, 2, 3, 4

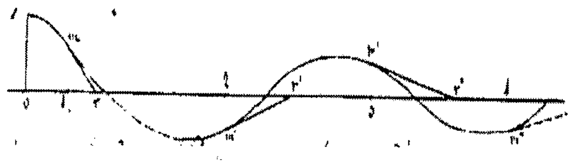

**fig.3 An exponential decay of diffusion in Fourier's Manuscript 1807**

## 3. THE THEORETICAL CONTRARIETIES TO FOURIER

### 3.1. Lagrange, Fourier and Poisson on the trigonometric series.

Riemann studies the history of research on Fourier series up to then (*Geschichte der Frage über die Darstellbarkeit einer willkührlich gegebenen Function durch eine trigonometrische Reihe*, [83, pp.4-17].) We cite one paragraph of his interesting description from the view of mathematical history as follows :

> ($\Leftarrow$) When Fourier submitted his first work to the Academy française[6] (21, Dec., 1807) on the heat, representing a completely arbitrary (graphically), given functions with the trigonometric series, at first, gray-haired Lagrange[7] irritates so much, however, refuses flatly. The paper is called now being belonged to the Arcive of the Parisian Academy française. (id. According to Mr. Professor Dirichlet's oral presentation.) Therefore, after Poisson inspects carefully through the paper,[8] promptly argues that in the paper of Lagrange, there is a paragraph on the vibration of string, where Fourier may have discovered the descriptive method.[9] To refuse this defect of the statement telling clearly on the rivalry relation between Fourier and Poisson, we would like to back to the Lagrange's papers, so we can reach the event in the Academy nothing have been clear yet. [83, p.10] (trans. mine.)

Riemann cites exactly the French original as follows :

($\Leftarrow$) In fact, a paragraph cited by Poisson is the expression :

$$y = 2\int Y \sin X\pi dX \ \sin x\pi + 2\int Y \sin 2X\pi dX \ \sin 2x\pi + \cdots + 2\int Y \sin nX\pi dX \ \sin nx\pi, \quad (1)$$

> So, If $x = X$, then $y = Y$, and $Y$ is the ordinate confronting to the abscissa $X$. This formula doesn't coincide with the Fourier's series[10] ; there is sufficiently the capability of some mistake; however, it is only a simple outlook, because Lagrange uses $\int dx$ as the integral symbol. Today, it is to be used by $\sum \Delta X$. When we inspect through his papers, it is beyond believable that he expresses a completely arbitrary function by series expansion with infinite sins. [83, pp.10-11] (trans. mine.)

Lagrange had stated (1) in his paper of the motion of sound in 1762-65. [45, p.553]

[6]($\Downarrow$) i.e. French Academy.
[7]($\Downarrow$) Lagrange was then seventy-one years old.
[8]id.
[9]id.
[10]($\Downarrow$) This means two interpretations : one means the series by Fourier, the other today's conventionally used nomenclature : 'the Fourier series'. Judging from Riemann's young days, in 1867, this may mean the former. In generally, the trigonometric series is used then.

### 3.2. **The trials to seek the mathematical rigours on heat theories.**

Poisson [64] traces Fourier's work of heat theory, from the another point of view. Poisson emphasizes, in the head paragraph of his paper [64], that although he totally takes the different approaches to formulate the heat differential equations or to solove the various problems or to deduce the solutions from them, the results by Poisson are coincident with Fourier's. Poisson says as follows in the top page of [64] :

> The question, which I propose to research, have been the subject of the prize proposed by the first class of the Institute, and won by Fourier at the beginning of 1812. The piece prized is reserved at the secretariat, where, it is permitted to look through : I will take care of, through this Memoire, to cite the principle result which Mr. Fourier have obtained before me ; and I dare to say at first, in all the particular problems which we have taken the one and the another for examples, and which being naturally indicated in this material, the formulae of my Memoire coincides with that this piece includes. However, *just only that there is common between our two oeuvres* ; because, it were to formulate the differential equations of the motion of the heat, or it were to solve them and deduce the definitive solution of each problem, *I am using the entirely different methods from that Mr. Fourier is tracing.* [64, pp.1-2] (trans. and italics mine.)

Poisson [64] considers the proving on the convergence of series of periodic quantities by Lagrange and Fourier as the manner lacking the exactitude and vigorousness, and wants to make up to it. Poisson proposes the different and complex type of heat equation with Fourier's. For example, we assume that interior ray extends to sensible distance, which forces of heat may affect the phenomena, the terms of series between before and after should be differente.

We remark that Fourier's integral problems are handled in the scope on the infinite solid in Fourier 1822 [22]. We must pay attention to that these considerations have been capable on the continuum theory.

### 3.3. **Trigonometric series.**

Poisson shows his trigonometric series as the rivalry to Fourier as follows :

$$(14)_{PS7} \qquad f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \cos\ \alpha\ (x - x')\ f(x')\ d\alpha\ dx', \qquad (2)$$

Poisson says : of which Fourier enhanced the Analysis, or, at least, that he gave at the first time for the cases where we have $f(x) = f(-x)$ or $f(x) = -f(-x)$, and of which he has been easy to deduce the general formula. We show an example of Fourier's trigonometric series as follows :

$$(p) \qquad r\ f(x) = \frac{1}{2}\int F(x)dx$$
$$+ \quad \cos\ x\int F(x)\ \cos\ x\ dx + \cos\ 2x\int F(x)\ \cos\ 2x\ dx + \cdots$$
$$+ \quad \sin\ x\int F(x)\ \sin\ x\ dx + \sin\ 2x\int F(x)\ \sin\ 2x\ dx + \cdots$$

[22, §233, p.230] or [23, §233, p.256].

Poincaré 1895 [78] proves the existence of the function satisfying the Dirichlet condition :

*Thèoréme. - Si une fonction $f(x)$ satisfait à la condition de Dirichlet dans l'intervalle $(-\pi, \pi)$, elle pourra être représentée dans ce même intervalle par une sêrie de Fourier, c'est-à-dire que l'on aura :*

$$\pi f(x) = \frac{1}{2}\int_{-\pi}^{\pi} f(x)\ dx + \sum \cos mx \int_{-\pi}^{\pi} f(x) \cos mx\ dx + \sum \sin mx \int_{-\pi}^{\pi} f(x) \sin mx\ dx$$

[78, p.57, §38] (cf. Table 3.)

## 4. Confusions and unify on continuum theory

The hysico-mathematicians are must construct at first the physical structure, then allpies the mathematical concept on it. The former is necessary to fit with the actual phenomena. Arago 1829 [1] seeks to separate these items to Navier 1829 [53] in the current of dispute with Poisson and Arago. This is comes from the word what-Navier-called *l'une sur l'autre*, he fails to explain exactly it, and since then, his theories and the equations are neglected up to the top of the 20th century. We consider that the confusions and unify are as follows :

- Poisson and Fourier discuss on the handling of the De Gua's theorem into the transcendental equations. Without clear explanation, Fourier passed away in 1830. cf. (fig.4)
- On the attraction and replusion of molecule, Navier depends on Fourier's principle of heat molecule. The then hysico-mathematicians had little evaluated Navier until the top of the 20th century. For formulation of heat motion in the fluid, Fourier cites not Navier's fluid equations, but Euler's fluid equations.
- The hydrodynamists like Navier, Poisson, Cauchy are propose the wave equations in the elasticity, and the last two hydrodynamists proposes the total equations in unity on the continuum.
- On the formulation of heat motion in the fluid, Fourier had submitted this paper, however, until his death, he has not published it, in which he seems to aim the unity of hydro- and thermodynamics, however, he has given up it.

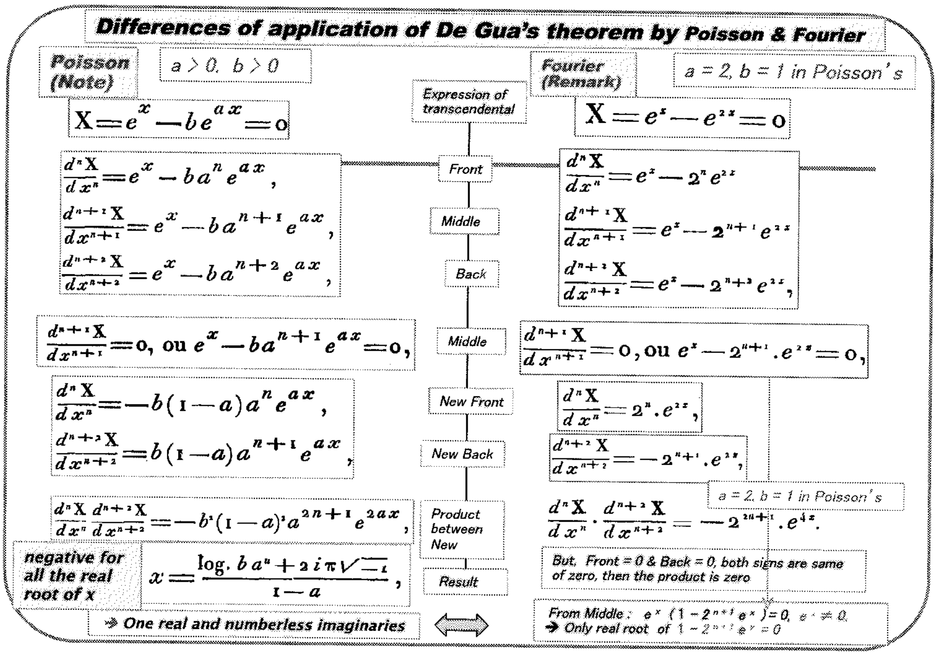


**fig.4 Difference of applying the De Gua's theorem into the transcendental between Poisson and Fourier**

## 4.1. A comment on continuum by Duhamel.

Duhamel 1829 [15] points out the theory of continuum from the viewpoint of scientific history, citing from the Poisson's paper in the argument with Navier on the nonsense of Navier's null action in nature.

> (⇐) Up to now, the reserchers have considered the corps of the nature as continue, it makes illusion to this regards, however, partly because this hypothesis simplify the calcul, and partly because they think that it gives a sufficient approximation. Mr. Poisson think that this hypothesis isn't never admissible, and justify his opinion with following considerations.
>
> > When a corps, say, is in its natural state, namely, when it isn't compressed with any force, when it is placed in the vacume, and when we make abstract of its weight, not only any molecule is in equilibrium in the interior and its surface, but also, we see more over, in this Memoire, the resultant of molecular actions is separately zero of two opposite sides od each small part of the corps. In this state, the distance which separate the molecules must be such that this condition were replaced, in having regard to their mutual attraction and the caloric repulsion which we take also among the molecular actions. However the corps is hard or something solid, the

> force which opposes the separation of their parties is zero or doesn't exist in the state of which we discuss. It doesn't begin the existence that when we seek to effectuate this separation, and when we change only a few distance of the molecules. Namely, if we explain this force with a integral, it gets to as its value being zero in the natural state of corps, this will be so even if after the variation of the molecular distances, so that, the corps will opposite any resistance to the separatiopn of its parties ; this is what will be nonsense. It results from here, that the sum which explain the total action of a series of disjoint molecules can't convert the sum instead of the definite integral ; this is what holds in the nature of the *function of distances* which represent the action of each molecule. The molecular force, of which we will find the expression in the §1 of this Memoire, is calculated according to this principle, and reduced at least in the simplest form of which it were susceptible.
>
> We explain afterward how he do with Mr. Poisson obtain the same equation with Navier has made known in 1821, with talking the molecular actions, and in considering the corps as continue. This method inspecting the molecular actions is originally due to Laplace, who has deduced from this a nice theory of capillary action. Mr. Navier has obtained afterward the nice idea to deduce the theory of elastic solid ; however, both of the mathematicians have supposed the molecules of adjacent corps, and Poisson is the first of coincidence with calculations with the physical structures. In addition to, although the hypotheses of continuum theory have been actually so inexact, however, have played big roles in the science, In the roles, have played, the theories by Mr. Laplace have welcomed by the researchers. This observation on the molecular activities, in the bulk of special problems, above all, in theory of the elastic bodies, it has the very countless merits to have to sweep out the all special hypotheses. Mr. Poisson emphasizes the merit of this method ; we will reproduce textually this passage from his Mmoire. [15, pp.98-99] (trans. and italics mine.)

Poisson explains the function $f(r)$ of distance between the two molecules. If using the integral, then as follows :

$$K = \frac{2\pi}{3}\int_0^\infty \frac{r^3}{\alpha^6} f(r)dr, \quad k = \frac{2\pi}{15}\int_0^\infty \frac{r^5}{\alpha^6} d.\frac{1}{r} f(r), \tag{3}$$

> en multipliant sous les signes $\sum$ par $\frac{dr}{\alpha}$, et remplaçant ces signes par ceux de l'intégration. Or, si l'on intègre par partie, et si l'on fait attention que $f(r)$ est nulle aux deux limites, il en résultera
>
> $$k = -\frac{2\pi}{3}\int_0^\infty \frac{r^3}{\alpha^6} f(r)dr = -K \tag{4}$$
>
> ce qui montre que la quantité $K$ étant nulle, on aurait aussi $k = 0$. [72, pp.398-399, §14]

(⇐) This equation becomes the cause of important remark. The sums $\sum$ in the no. 6 expressed by $K$ and $k$, which we can't transform into the integral, while the variable $r$ increases in each of them with very small differences equal to $\alpha$. Therefore, if this transformation would be capable, $k$ and $K$ become zero syncronously. From here, it would result

- that after changing of the form of corps, the forces $P$, $Q$, $R$ would be the same zeros as before,
- and that the given forces operating on the corps could not make equilibrium.

These are inadmissible. [72, pp.398-399, §14] (trans. mine.)

We show Poisson's logic as follows : Poisson proposes the two constants must be with the sum :

$$\frac{2\pi}{3}\sum \frac{r^3}{\alpha^5} f(r) \equiv K, \quad \frac{2\pi}{15}\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}f(r)}{dr} \equiv k.$$

When begining to perform the integral instead of the sum with $k$,

$$d\Big(\frac{1}{r}f(r)\Big) = \frac{1}{r}f'(r)dr - f(r)\Big(\frac{1}{r^2}\Big)dr$$

$$k = \frac{2\pi}{15}\int_0^\infty \frac{r^5}{\alpha^6} d\Big(\frac{1}{rf(r)}\Big) = \frac{2\pi}{15}\int_0^\infty \frac{r^4}{\alpha^6} f'(r)dr - \frac{2\pi}{15}\int_0^\infty \frac{r^3}{\alpha^6} f(r)dr \equiv I - \frac{1}{5}K.$$

Now, if we take the integral by part, and if we pay attention that $f(r)$ is zero at the two limits, then we get the next

$$I = -\frac{8\pi}{15}\int_0^\infty \frac{r^3}{\alpha^6} f(r)dr = -\frac{4}{5}K \quad \Rightarrow \quad k = -K.$$

Navier points out Poisson's assumption that if $f(r)$ is zero at the two limits. There are many functions which don't take such values. Poisson pays attention to all the case in probability.

## 4.2. **Attraction and repulsion.**

Here, we show one of Fourier's contexts which Navier depends on and esteems as the authority of hysico-mathematicians.

> (⇐) ¶54. The equilibre which keeps in the interior of a solid mass between the *repulsive force* due to heat and the molecular *attraction* is stable ; namely, which restablish by iiself, when it troubles by an accidental cause. If the molecules are places in the distance which is convenient to the equilibre, and if an exterior force make this distance without the temperature changes by the heat, the effect of *attraction* begins surpass it and makes the molecules at the initial position, after a multitude of oscilation which becomes more and more insensible. A resemble effect operates when a mechanic cause shortens the initial distance of the molecules ; this is the origin of sonic or flexible vibration of corps and of all the effect of elasticity. [22, pp.31-2] (trans. mine.)

TABLE 1. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" were fixed. ($HD$ : hydrodynamics, $N$ : non-linear, g.d : grad.div, $C$ : $\frac{\Delta}{gr.dv}$ in elastic or fluid. $\Delta$ : tensor coefficient of the main axis in Laplacian.)

| no | name/ prob | the kinetic equations | $\Delta$ | g.d | C |
|---|---|---|---|---|---|
| 1 N | Euler (1752 -55)[17] fluid | $\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases}$ $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$ | | | |
| 2 | Lagrange (1782)[46] fluid | $\begin{cases} \Delta\left[\left(\frac{d^2x}{dt^2} + X\right)\frac{\partial x}{\partial a} + \left(\frac{d^2y}{dt^2} + Y\right)\frac{\partial y}{\partial a} + \left(\frac{d^2z}{dt^2} + Z\right)\frac{\partial z}{\partial a}\right] - \frac{\partial \lambda}{\partial a} = 0, \\ \Delta\left[\left(\frac{d^2x}{dt^2} + X\right)\frac{\partial x}{\partial b} + \left(\frac{d^2y}{dt^2} + Y\right)\frac{\partial y}{\partial b} + \left(\frac{d^2z}{dt^2} + Z\right)\frac{\partial z}{\partial b}\right] - \frac{\partial \lambda}{\partial b} = 0, \\ \Delta\left[\left(\frac{d^2x}{dt^2} + X\right)\frac{\partial x}{\partial c} + \left(\frac{d^2y}{dt^2} + Y\right)\frac{\partial y}{\partial c} + \left(\frac{d^2z}{dt^2} + Z\right)\frac{\partial z}{\partial c}\right] - \frac{\partial \lambda}{\partial c} = 0 \end{cases}$ where, $\mathbf{a} = (a, b, c)$ : position on $t = 0$, $\mathbf{X} = (x,\ y,\ z)$ : position on $t = t$, $\mathbf{X} = (X,\ Y,\ Z)$ : outer force, $\Delta$ : density, $\lambda$ : pressure. Today, these are described as follows : $\rho\sum_{j=1}^{3}\frac{\partial x_j}{\partial a_i}\left(\frac{\partial^2 x_j}{\partial t^2} - K_j\right) = -\frac{\partial p}{\partial a_i},\ (i = 1, 2, 3),$ | | | |
| 3 | Navier (1827) [51] elastic sol. | $(6\text{-}1)_{N^e}$ $\begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where $\Pi$ is density of the solid, $g$ is acceleration of gravity. | $\varepsilon$ | $2\varepsilon$ | $\frac{1}{2}$ |
| 4 | Navier (1827) [52] fluid | $\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) \\ \qquad -\frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) \\ \qquad -\frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) \\ \qquad -\frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w \end{cases}$ | $\varepsilon$ | $2\varepsilon$ | $\frac{1}{2}$ |
| 5 | Poisson ('31)[73] elastic in gen. | $\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$ | $\frac{a^2}{3}$ | $\frac{2a^2}{3}$ | $\frac{1}{2}$ |
| 6 | Poisson ('31) [73] fluid in general eq. | $\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \\ \quad +\frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) \\ \quad +\frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) \\ \quad +\frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \end{cases}$ $\Rightarrow \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ $\begin{cases} \text{where ,} \\ \varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \\ \beta \equiv \alpha(K+k) \end{cases}$ | $\beta$ | $\frac{\beta}{3}$ | 3 |

TABLE 2. (Continued from Table 1.) The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" were fixed.

| no | name/ prob | the kinetic equations | $\Delta$ | g.d | C |
|---|---|---|---|---|---|
| 7 | Stokes ('49)[87] fluid | $(12)_S$ $\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0, \\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0, \\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0. \end{cases}$ | $\mu$ | $\frac{\mu}{3}$ | 3 |
| 8 N | Prandtl (1905) [79], HD | $\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)+\nabla(V+p)=k\nabla^2 v, \quad \mathrm{div}\,\mathbf{v}=0$ | $k$ | | |
| 9 N | Prandtl (1934) [80], HD | $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$<br>$=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),$<br>for incompressible, it is simplified as follows :<br>$\mathrm{div}\,\mathbf{w}=0, \quad \frac{D\mathbf{w}}{dt}=g-\frac{1}{\rho}\mathrm{grad}\,p+\nu\Delta\mathbf{w}$ | $\nu$ | $\frac{\nu}{3}$ | 3 |
| 10 | Saint-Venant [84] | ('43) fluid. He didn't describe the equations in [84]. , however his tensor is in Table ?? (entry no.4) | $\varepsilon$ | $\frac{\varepsilon}{3}$ | 3 |
| 11 | Stokes ('49)[87] fluid | $(12)_S$ $\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0, \\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0, \\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0. \end{cases}$ | $\mu$ | $\frac{\mu}{3}$ | 3 |
| 12 | Maxwell ('65-66) [50] HD | $\begin{cases} \rho\frac{\partial u}{\partial t}+\frac{dp}{dx}-C_M\left[\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+\frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right]=\rho X, \\ \rho\frac{\partial v}{\partial t}+\frac{dp}{dy}-C_M\left[\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+\frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right]=\rho Y, \\ \rho\frac{\partial w}{\partial t}+\frac{dp}{dz}-C_M\left[\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}+\frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right]=\rho Z \end{cases}$<br>where, $C_M \equiv \frac{pM}{6k\rho\Theta_2}$ | $C_M$ | $\frac{C}{3}$ | 3 |
| 13 | Kirchhoff ('76)[37] HD | $\begin{cases} \mu\frac{du}{dt}+\frac{\partial}{\partial x}-C_K\left[\Delta u+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\mu X, \\ \mu\frac{dv}{dt}+\frac{\partial p}{\partial y}-C_K\left[\Delta v+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\mu Y, \\ \mu\frac{dw}{dt}+\frac{\partial p}{\partial z}-C_K\left[\Delta z+\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\mu Z, \end{cases}$<br>$\begin{cases} \frac{1}{\mu}\frac{d\mu}{dt}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$ | $C_K$ | $\frac{\Delta}{3}$ | 3 |
| 14 N | Rayleigh ('83)[82] HD | $\begin{cases} \frac{1}{\rho}\frac{dp}{dx}=-\frac{du}{dt}+\nu\nabla^2 u-u\frac{du}{dx}-v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy}=-\frac{dv}{dt}+\nu\nabla^2 v-u\frac{dv}{dx}-v\frac{dv}{dy} \end{cases}, \quad \frac{du}{dx}+\frac{dv}{dy}=0$ | $\nu$ | | |
| 15 | Boltzmann ('95)[4] HD | $(221)_B$ $\begin{cases} \rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}-\mathcal{R}\left[\Delta u+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\rho X, \\ \rho\frac{\partial v}{\partial t}+\frac{\partial p}{\partial y}-\mathcal{R}\left[\Delta v+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\rho Y, \\ \rho\frac{\partial w}{\partial t}+\frac{\partial p}{\partial z}-\mathcal{R}\left[\Delta w+\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\rho Z \end{cases}$ | $\mathcal{R}$ | $\frac{\mathcal{R}}{3}$ | 3 |
| 16 N | Prandtl (1905) [79], HD | $\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)+\nabla(V+p)=k\nabla^2 v, \quad \mathrm{div}\,\mathbf{v}=0$ | $k$ | | |
| 17 N | Prandtl (1934) [80], HD | $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$<br>$=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),$<br>for incompressible, it is simplified as follows :<br>$\mathrm{div}\,\mathbf{w}=0, \quad \frac{D\mathbf{w}}{dt}=g-\frac{1}{\rho}\mathrm{grad}\,p+\nu\Delta\mathbf{w}$ | $\nu$ | $\frac{\nu}{3}$ | 3 |

## 5. Fourier's heat equation of motion in fluid

Fourier explains the motion of the heat in the interior of solid. The difference is that determines its increment of the temperature during an instant with only the transfer of heat quantity, which is the most different method with Poisson's method :

$$K dydz\ d\Big(\frac{dv}{dx}\Big)dt + K dxdz\ d\Big(\frac{dv}{dy}\Big)dt + K dxdy\ d\Big(\frac{dv}{dz}\Big)dt$$
$$\Rightarrow \quad K dxdydz\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big)dt$$

then finally he gets :

$$(d)_{F2.5} \qquad \frac{du}{dt} = \frac{K}{C.D}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \tag{5}$$

where, K internal conductibility, C capacity, D density of the substance. [22, p.102] We think that Fourier's deductive method is very diffuse style and simpler than Poisson's inductive method described over 10 pages in original [77], we show his point below in § 6.1. Fourier doesn't show the pricise deduction of the heat equation (5), while Poisson takes 9 pages to descrive it from §44 to §50. We think Poisson's contribution on the physical mathematics such as the fluid dynamics and the heat theory including the trigonometric series is great.

Fourier esteems Euler's fluid dynamic equations, saying in the preface of "The analysis of the heat motion in the fluid." We cite Fourier's English translated paper as follows :

> To solve this, we must consider, a given space interior of mass, for example, by the volume of a rectangular prism composed of six sides, of which the position is given. We investigate all the successive alterations which the quality of heat contained in the space of prism obeys. This quantity alternates instantly and constantly, and becomes very different by the two things. One is the property, the molecules of fluid have, to communicate their heat with sufficiently near molecules, when the temperatures are not equal.
>
> The question is reduced into to calculate separately : the heat receiving from the space of prism due to the communication and the heat receiving from the space due to the motion of molecules.
>
> We know the analytic expression of communicated heat, and the first point of the question is plainly cleared. The rest is the calculation of transported heat : it depend on only the velocity of molecules and the direction which they take in their motion. [28, pp.507-514.]. (trans. mine.)

Fourier combines heat theory with the Euler's equation of incompressible fluid dynamics and proposes the equation of heat motion in fluid in 1820, however, this paper was published in 1833 after 13 years, it was after 3 years since Fourier passed away. Fourier seems to have been doutful to publish it in life. Here, $\varepsilon$ is the variable density and $\theta$ is the variable temperature of the molecule respectively. $K$ : proper conductance of mass, $C$ : the constant of specific heat, $h$ : the constant determining dilatation, $e$ : density at

$\theta = 0.$

$$\begin{cases} \frac{1}{\varepsilon}\frac{dp}{dx} + \frac{d\alpha}{dt} + \alpha\frac{d\alpha}{dx} + \beta\frac{d\alpha}{dy} + \gamma\frac{d\alpha}{dz} - X = 0, \\ \frac{1}{\varepsilon}\frac{dp}{dy} + \frac{d\beta}{dt} + \alpha\frac{d\beta}{dx} + \beta\frac{d\beta}{dy} + \gamma\frac{d\beta}{dz} - Y = 0, \\ \frac{1}{\varepsilon}\frac{dp}{dz} + \frac{d\gamma}{dt} + \alpha\frac{d\gamma}{dx} + \beta\frac{d\gamma}{dy} + \gamma\frac{d\gamma}{dz} - Z = 0. \\ \frac{d\varepsilon}{dt} + \frac{d}{dx}(\varepsilon\alpha) + \frac{d}{dy}(\varepsilon\beta) + \frac{d}{dz}(\varepsilon\gamma) = 0, \quad \varepsilon = e(1 + h\theta). \\ \frac{d\theta}{dt} = \frac{K}{C}\left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2}\right) - \left[\frac{d}{dx}(\alpha\theta) + \frac{d}{dy}(\beta\theta) + \frac{d}{dz}(\gamma\theta)\right]. \end{cases}$$

where, $\alpha$, $\beta$, $\gamma$, $p$, $\varepsilon$, $\theta$ are the function of $x$, $y$, $z$, $t$. $X$, $Y$, $Z$ are the outer forces. We think, Fourier seems to feel an inferiority complex to the fluid dynamics by Euler and he divers the Euler equation as the transport equation from Euler 1755 [17]. (cf. Table 1.)

## 6. Poisson's paradigm of universal truth on the definite integral

Poisson mentions the universality of the method to solve the differential equations. Poisson attacks the definite integral by Euler and Laplace, and Fourier's analytical theory of heat, and manages to construct universal truth in the paradigms.

One of the paradigms is made by Euler and Laplace. Laplace succeeds to Euler and states the passage from real to imaginary or reciprocal passage between two, which we mention in below.

The other contradictory problem is Fourier's application of De Gua. The diversion is Fourier's essential tool for the analytical theory of heat.

Dirichlet calls these passages a sort of *singularity of passage* from the finite to the infinite. cf. Chapter 1. We think that Poisson's strategy is to destruct both paradigms and make his own paradigm to establish the univarsal truth between mathematics and physics.

### 6.1. The deduction of heat equations by Poisson.

Poisson deduces his heat equations of the motion in interior of solid corps or liquid, from only §44-50. These are more precise than Fourier's, though their result is the same. Poisson's method is based on the hypothesis of molecular radiation. It may come from the fluid dynamics. For Poisson, the common method between the fluid dynamics and the heat analysis is molecular analysis. While in the fluid dynamics, the function of the distance is $f(r)$, in the heat theory, the corresponding function is the function of the distance, both the temperatures and both the coordinates, which is the expression (8) introduced in **§45**. We introduce the gist of the Poisson's molecular analysis on heat from **§44** to **§50**, which are the Poisson's sales point in rivalry to Fourier as follows :

**§44.**

There is always the heat in motion in all the corps, even when of all their points is invariable,

- were each point would have a particular temperature,
- were its would have all a same temperature.

However, the expression *motion of the heat* is taken here, in the another sense ; it signifies the variation of temperature which holds from an instant to the other in a corps which is heated or is cooled ; and the velocity of this motion, in each point of the corps, is the primary differential coefficient of the temperature with respect to the time.

I will call $A$ the corps solid or liquid, homogeneous or heterogeneous, in which we are going to consider the motion of the heat. Let

- $M$ a certain point of $A$,
- and $m$ a particle of this corps, of insensible magnitude (no. 7),
- and take the point $M$.

At the end of a certain time $t$,

- designate with $x,\ y,\ z$, the three rectangular coordinates of $M$,
- with $v$ the volume of $m$,
- and with $\rho$ its density,

so that we have $m = v\rho$. Let also, at the same instant, $u$ the temperature and $\mho$ [11] the velocity of motion of the heat which responds to the point $M$.

The quantity $u$ will be a function of $t,\ x,\ y,\ z$, dependent on an equation in the partial differences with respect to these four variables, which it is the problem to form. If $A$ is a corps solid, and which we make neglect its small dilations, positive or negative, products with the variations of $u$ relative to time, the coordinates $x,\ y,\ z$, according to independent of $t$, and we will have simply, $\mho = \frac{du}{dt}$.

- If in contrast, we have regard to small displacement of the point $M$ caused from these dilations,
- or also, if $A$ is a fluid in which the integrality of temperature, or all other cause, hold to the motions of its molecules,

then the coordinates $x,\ y,\ z$, will be the function of $t$ ; and then we will have with the known rules of the differentiation of functions made of functions,[12]

$$(1)_{PS4} \qquad \mho = \frac{du}{dt} + \frac{dx}{dt}\frac{du}{dx} + \frac{dy}{dt}\frac{du}{dy} + \frac{dz}{dt}\frac{du}{dz}\ ; \qquad (6)$$

where, expression in which $\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}$, will be the components of the velocities at the point $M$, parallel to the axes $x,\ y,\ z$.

The unknown $u$ will be the only that it will need to determine, for recognition completely of the calorific state of the corps $A$ at a certain instant. Suppose that we divide this corps into two parts $B$ and $B'$, with a certain surface, traced in its interior. Let $\omega$ an element of this surface (no. 9) containing the point $M$, there will be continuously, crosswise $\omega$, a *flux of heat* sensible to that of the radiating heat which holds crosswise the element of the surface of $A$, and that I will represent with $\Gamma\ \omega\ dt$ during the instant $dt$, of manner which this product, positive or negative, were the excess of the heat which traverses the $\omega$ in passing from $B$ in $B'$, during this instant, on that which traverse in the same time, in passing from $B'$ into the $B$. The coefficient $\Gamma$, or the flux of heat relative to the units of time and of surface. will depend on the material and of the temperature of $A$ at the point $M$, and of the direction of $\omega$ ; it will be important to determine it, in function of $t,\ x,\ y,\ z$, for each direction given with $\omega$. Hence, $u$ and $\Gamma$ will be the two unknown of the problem of which we will have to us occupy in this chapter. When the corps $A$ is obey to the influence of foci constants of heat all its parts arrive generally, after a certain time, to the variable temperatures of a point to another, however, independent of the time. In this stationary state of $A$, the velocity $\mho$ is zero in all the point ; however, the flux of heat

[11]($\Downarrow$) We use $\mho$, because, in origin, Poisson uses the vertical type of $\propto$ like the opened shape in upper of the numerical letter 8, however, this exact type isn't in our LaTex font system.

[12]($\Downarrow$) sic. The function is repeated.

$\Gamma$ exists still, and merely its value is independent of $t$, like that of $u$.

## §45.

Let $M'$ a second point of $A$ very near to $M$, and $m'$ a particle of $A$ of insensible magnitude, like $m$ which will contain $M'$. At the end of time $t$, we call $x'$, $y'$, $z'$, the coordinates of $M'$ in relating to same axes with $x$, $y$, $z$, and designate with $u'$ the temperature of $m'$ ; also let $r$ the distance $MM'$.

According to the general hypothesis on which the mathematical theory of the heat (no. 7) is based, there will be a continuous exchange of heat between $m$ and $m'$. I will represent with $\delta$ the augmentation of heat which will result then for $m$ during the instant $dt$, namely, the excess positive or negative, during this instant,

- of the heat emitted from $m'$ and absorbed with $m$,
- over the heat emitted from $m$ and absorbed with $m'$.

It will be able to suppose this excess proportional to product $m\ m'\ dt$, or to $v\ v'\ \rho\ \rho' dt$, in calling $v'$ and $\rho'$ the volume and the density of $m'$, so that we would have $m' = v'\ \rho'$, as we have already $m = v\ \rho$. It will be zero in the case of $u' = u$, and same sign with the difference $u' - u$, when it won't be zero ; in the vacuum, it will come in the reverse ratio of the square of $r$ ; and generally its value will be the form

$$(2)_{PS4} \qquad \delta = \frac{v\ v'}{r^2}\ R\ (u' - u)\ dt, \tag{7}$$

where, in designating with $R$ a positive coefficient, in which we contain the factor $\rho\ \rho'$, which will decrease very rapidly for the values increasing with $r$, which will be also able to depend on materials and the temperatures of $m$ and $m'$, and will vary with the direction of $MM'$, when the absorption of the heat won't be the same in all direction around of $M$.

In the supposition the most general, $R$ will be hence a function of $r$, $u$, $u'$, and the coordinates of $M$ and $M'$ ; so that we will have

$$R = \Phi\ (r,\ u,\ u',\ x,\ y,\ z,\ x',\ y',\ z'). \tag{8}$$

However, if we call $\delta'$ the dimension of heat of $m'$ during the instant $dt$, causing the exchange between $m$ and $m'$, we will have evidently $\delta' = -\delta$ ; in addition, the value of $\delta'$ will come to be deduced from that of $\delta$ with the permutation of quantities relative to the one of the points $M$ and $M'$, and the analogous quantities which respond to the other ; in consequence, it will need that the function $\Phi$ were symmetric with respect to $u$ and $u'$, $x$ and $x'$, $y$ and $y'$, $z$ and $z'$.

The corps $A$ being a solid or a liquid, this function $\Phi$ will vary very rapidly with $r$ and will be insensible or zero, as long as $r$ will have arrived at a very small magnitude. I will designate this limit with $l$, so that this function $\Phi$ were zero, as long as we will have $r > l$ or merely $r = l$. This segment $l$ will be hence very small, however, of the sensible magnitude and measurable (no. 41), and in consequence, extremely greater with relation to the dimensions of $m$ and $m'$.

## §46.

The total augmentation of heat of $m$ during the instant $dt$ will be the sum of values of $\delta$, extended to all the point $M'$ of which the distance at the point $M$ is smaller than $l$. I will indicate a such sum with the characteristic $\Sigma$. The factor $v\ dt$ being common to all the value of $\delta$, their sum will be

$$v\ dt \sum \frac{R}{r^2}\ (u' - u)\ v'. \tag{9}$$

However, during the instance $dt$, the temperature of $m$ augments with $\mho\ dt$ ; if hence, we call $c$ its specific heat, $c\ v\ \mho\ dt$ will be also its augmentation of heat during this instant ; hence in suppressing the common factor $v\ dt$, we will have

$$(3)_{PS4} \qquad c\,\mho = \sum \frac{R}{r^2}\ (u' - u)\ v'. \tag{10}$$

for the equation of motion of the heat equally applicable to a corps solid and to a liquid, in substituting the convenient expression with $\mho$.

The sum $\sum$ contained in this equation, doesn't depend in effect, merely on the calorific state of $m$ and of the particles surrounding with $A$, which exists at the end of the time $t$, and in any manner of change which would be able to hold the next instant ; so that it wouldn't be necessary to the heat, like the mathematicians[a] have considered, a particular equation for the motion of the heat in the liquids[b], distinct from one which responds to corps solids heterogeneous, and which had been given since long ago.

[a]($\Downarrow$) F. geometricians. Now, it means mathematician.
[b]($\Downarrow$) Poisson may cite as the mathematician Fourier [28].

The value of a sum $\sum$ relative to the particles of insensible magnitude, such that the preceding, can be explained with a series of which the primary term is a integral taken between the same limits which this sum, and of which the other preceding terms following the dimensions of these particles, raised to the increasing power. These dimensions being insensible with hypothesis, it is followed that the series is, in general, extremely convergent, and may be reduced to its primary term. Hence, in designating with $d\ v'$ the differential element of the volume of $A$, which responds to the point $M'$, we will have, without appreciative error,

$$\sum \frac{R}{r^2}\ (u' - u)\ v' = \int \frac{R}{r^2}\ (u' - u)\ dv'\ ;$$

The integral is extending to all the element $dv'$, of which the distance $r$ at the point $M$ is smaller than $l$.

In effect, I remarked in other occasions which the reduction of a sum to a integral is no more permitted in a certain case which is presented, for instance, in the calculation of molecular forces ; however, for that this exception would hold, it needs that the function of which we are going to sum the values, varies very rapidly and change the sign between the limits of this sum ; hence, here the coefficient $R$ vary well in effect very rapidly with the variable $r$, however, without never change of sign ; and for this reason, the exception of which it is important isn't to be afraid. In all the calculation of quantities of heat which result of exchange between the particles of a corps, of insensible magnitude, we will be able to decompose immediately its volume in elements infinitely smaller, and replace the sum with the integrals, as if this corps being would be formed of a material, contained and not of the disjoint molecules, separated with the pores or vacant space.

## §47.

Of the point $M$ as center and a radius equal to the linear unit, we describe a spherical surface ; were $ds$ the differential element of this surface, to which gets, the radius of which the direction is that of $MM'$, we will have

$$dv' = r^2 \ dr \ ds \ ;$$

and according to the value of the sum $\sum$, the equation (10) will turn out

$$(4)_{PS4} \qquad c \, \frac{du}{dt} = \iint R \ (u' - u) \ dr \ ds \ ; \tag{11}$$

We put here, for abridgement, $\frac{du}{dt}$, instead of Ʊ ; however, we will remember that this differential coefficient needs to be taken with relation to $t$ and to all this that depend ; so that it needs to replace $\frac{du}{dt}$ with the formula (6), when the coordinates $x, \ y, \ z,$ of the point $M$ will vary with the time.

The limit relative to $r$ of the integral contains in this equation (11) won't be the same, according to the distance of the point $M$ to the surface of $A$ will surpass $l$ or will be shorter than this small segment. In this chapter we will suppose that this were the primary case which holds ; the integral relative to $r$ will come to be hence taken from $r = 0$ to $r = l$, in all the direction around $M$ ; we will be able hence to describe the equation (11) under the form

$$(5)_{PS4} \qquad c \, \frac{du}{dt} = \int_0^l \Big[ \int R \ (u' - u) \ ds \Big] \ dr \ ; \tag{12}$$

where, the integral in respecting to $ds$ will come to be extended to all the element $ds$ from the spherical surface, and with the reduction in series, we will obtain easily the approximate value.

## §48.

For these things, I designate with $\alpha, \ \beta, \ \gamma,$ the angles which the segment $MM'$ makes with the parallels to the axes $x, \ y, \ z,$ traced through the point $M$. Because of $MM' = r$, then it will result

$$x' - x = r \ \cos \ \alpha, \qquad y' - y = r \ \cos \ \beta, \qquad z' - z = r \ \cos \ \gamma \ ;$$

and, according to the theory of Taylor, we will have

$$\begin{aligned} u' - u = \ & \frac{du}{dx} \ r \ \cos \ \alpha + \frac{du}{dy} \ r \ \cos \ \beta + \frac{du}{dz} \ r \ \cos \ \gamma \\ + \ & \frac{1}{2} \frac{d^2u}{dx^2} \ r^2 \ \cos^2 \ \alpha + \frac{1}{2} \frac{d^2u}{dy^2} \ r^2 \ \cos^2 \ \beta + \frac{1}{2} \frac{d^2u}{dz^2} \ r^2 \ \cos^2 \ \gamma \\ + \ & \frac{d^2u}{dx \ dy} \ r^2 \ \cos \ \alpha \ \cos \ \beta + \frac{d^2u}{dx \ dz} \ r^2 \ \cos \ \alpha \ \cos \ \gamma + \frac{d^2u}{dy \ dz} \ r^2 \ \cos \ \beta \ \cos \ \gamma \\ \cdots \ . \end{aligned}$$

If we develop similarly $R$ in accordance with the power and the products of $u' - u, \ x' - x, \ y' - y, \ z' - z,$ we will have also

$$R = V + \left(\frac{dR}{du'}\right) \ (u' - u) + \left(\frac{dR}{dx'}\right) \ (x' - x) + \left(\frac{dR}{dy'}\right) \ (y' - y) + \left(\frac{dR}{dz'}\right) \ (z' - z) + \cdots \ ;$$

where, the parentheses indicating here that it needs to put $u' = u$, $x' = x$, $y' = y$ , $z' = z$ according to the differentiation which supposes $r$ invariable, and $V$ designating this which comes at the same time from the function $\Phi$ of the (no. 45), so that we have

$$V = \Phi\ (r,\ u,\ u,\ x,\ y,\ z,\ x,\ y,\ z). \tag{13}$$

By means of these developments of $R$ and of $u' - u$, this one of product $\int R\ (u' - u)$ will be composed of terms of this form

$$H\ r^n\ \cos^i\ \alpha\ \cos^{i'}\ \beta\ \cos^{i''}\ \gamma\ ;$$

where, $H$ designating a coefficient independent of $\alpha$, $\beta$, $\gamma$, and the exponential $i$, $i'$, $i''$, being the number entire and positive which won't be zeros all the three to the times, and of which the exponent $n$ is the sum $i + i' + i''$. Hence in having regard to the limits of the integral relative to $ds$, we will have

$$\int \cos^i\ \alpha\ \cos^{i'}\ \beta\ \cos^{i''}\ \gamma\ ds = 0,$$

here all times which the one of the three numbers $i$, $i'$, $i''$, will be odd ; for then this integral will be composed of the elements which will be equal two by two and the contrary sign. When any of number $i$, $i'$, $i''$, won't be odd, the integral won't be zero ; the ordinary rules give the exact values, whatever these three number ; and with this manner, we will have

$$(6)_{PS4} \qquad R\ (u' - u) = H_2\ r^2 + H_4\ r^4 + H_6\ r^6 + \cdots\ ; \tag{14}$$

where, $H_2$, $H_4$, $H_6$, $\cdots$ , being the differential function of known form, in any of which the partial differences [13] of $u$ will be taken with respect to $x$, $y$, $z$, and are raised to the order marked with its inferior index.

For a temperature $u$ which would vary very rapidly, so that it would have the values very different in the extent of interior radiation, the coefficients $H_2$, $H_4$, $H_6$, $\cdots$ , would form a series very rapidly increasing, by reason of partial differences [14] of $u$ on which they depend. The series (14) would cease hence to be converged, though the smallness of $r^2$ ; however, this case doesn't hold in a point $M$ sufficiently separated, as we suppose it, of the surface of $A$ ; and we will be able, in consequence, to regard the series (14) as extremely convergent.

In stopping at its nth term, the equation in the partial differences [15] of the motion of the heat will be the order $2n$ ; however, its complete integral will include certain parties which will vary very rapidly, and that we be will be able to suppress for this reason, in the value of $u$, as a layperson to the question ; this one which will reduce always this value at the same degree of generality, whatever its degree of approximation, dependent on the terms of the series (14) which we will have conserved.

[13]($\Downarrow$) id. This mean the partial differentials.
[14]($\Downarrow$) id.
[15]($\Downarrow$) id.

This is here which we see successively, on a particular example, in which we will show also the influence which can have the sensible extent of the interior radiation on the value of $u$. However, to reduce the general equation of the motion of the heat to the simplest form, namely, to the form of an equation in the partial differences [a] of second order, also which we make ordinarily, we restrict the approximation to the primary term of the series (14) ; this is here which return to consider as insensible the extent of the radiation in the interior of corps solid and of liquid.

[a](⇓) id.

**§49.** (General equation of the motion of heat) [16]

In this hypothesis, we will stop the development of $R$ at the terms dependent on the square of $r$ exclusively. By reason of the system of $R$ in respect to $u$ and $u'$, $x$ and $x'$, $y$ and $y'$, $z$ and $z'$, and of this one which $V$ represents, we have evidently

$$\left(\frac{dR}{du'}\right)=\frac{1}{2}\frac{dV}{du},\quad \left(\frac{dR}{dx'}\right)=\frac{1}{2}\frac{dV}{dx},\quad \left(\frac{dR}{dy'}\right)=\frac{1}{2}\frac{dV}{dy},\quad \left(\frac{dR}{dz'}\right)=\frac{1}{2}\frac{dV}{dz}\ ;$$

then, it will result hence

$$R=V+\frac{1}{2}\frac{dV}{du}\,(u'-u)+\frac{1}{2}\frac{dV}{dx}\,(x'-x)+\frac{1}{2}\frac{dV}{dy}\,(y'-y)+\frac{1}{2}\frac{dV}{dz}\,(z'-z)\ ;$$

and of this value jointed to that of $u'-u$, we will conclude

$$\begin{aligned} H_2 &= \frac{1}{2}\Big[V\frac{d^2u}{dx^2}+\Big(\frac{dV}{du}\frac{du}{dx}+\frac{dV}{dx}\Big)\frac{du}{dx}\Big]\int\cos^2\alpha\ ds+\frac{1}{2}\Big[V\frac{d^2u}{dy^2}+\Big(\frac{dV}{du}\frac{du}{dy}+\frac{dV}{dy}\Big)\frac{du}{dy}\Big]\int\cos^2\beta\ ds\\ &+ \frac{1}{2}\Big[V\frac{d^2u}{dz^2}+\Big(\frac{dV}{du}\frac{du}{dz}+\frac{dV}{dz}\Big)\frac{du}{dz}\Big]\int\cos^2\gamma\ ds, \end{aligned}$$

or more simply

$$\begin{aligned} H_2 &= \frac{1}{2}\Big[V\frac{d^2u}{dx^2}+\frac{dV}{dx}\frac{du}{dx}\Big]\int\cos^2\alpha\ ds+\frac{1}{2}\Big[V\frac{d^2u}{dy^2}+\frac{dV}{dy}\frac{du}{dy}\Big]\int\cos^2\beta\ ds\\ &+ \frac{1}{2}\Big[V\frac{d^2u}{dz^2}+\frac{dV}{dz}\frac{du}{dz}\Big]\int\cos^2\gamma\ ds\ ; \end{aligned}$$

the partial differences [17] of $V$ with respect to $x$, $y$, $z$, being taken in considering $u$ as a function of these three coordinates, and without varying $r$.

We have additionally

$$\int\cos^2\alpha\ ds=\int\cos^2\beta\ ds=\int\cos^2\gamma\ ds.$$

Moreover, if we call $\psi$ the angle which makes the plane of the segment $MM'$ and of a parallel to the axis of $x$ traced through the point $M$, with a fixed plane traced through this parallel, we will have

$$ds=\sin\ \alpha\ d\alpha\ d\psi\ ;$$

[16](⇓) This article is the most frequently referred from other article, such as 52, 58, 64, 68, 70, **76**, 85, 89, 117, 119, 120, 137, **162**. (These are the article numbers, referred to the no. 49, and in the bold numbers, the another equations are expressed.)

[17](⇓) id.

and the integral relative to $ds$ will come to be extended to all the spherical surface, to which this element belongs, then it will result

$$\int \cos^2\alpha\ ds = \int_0^{\pi} \cos^2\alpha\ \sin\ \alpha\ d\alpha \int_0^{2\pi} d\psi = \frac{4\pi}{3}.$$

[18] Hence, in reducing the value of $\int R\ (u' - u)$ at the primary term $H_2\ r^2$ of the series (14), the equation (12) will come to be

$$\begin{aligned} c\frac{du}{dt} &= \frac{2\pi}{3}\Big(\frac{d^2u}{dx^2}\int_0^l V\ r^2\ dr + \frac{du}{dx}\int_0^l \frac{dV}{dx}\ r^2\ dr\Big) + \frac{2\pi}{3}\Big(\frac{d^2u}{dy^2}\int_0^l V\ r^2\ dr + \frac{du}{dy}\int_0^l \frac{dV}{dy}\ r^2\ dr\Big) \\ &+ \frac{2\pi}{3}\Big(\frac{d^2u}{dz^2}\int_0^l V\ r^2\ dr + \frac{du}{dz}\int_0^l \frac{dV}{dz}\ r^2\ dr\Big). \end{aligned} \tag{15}$$

The function $V$ being zero for all the value of $r$ longer than $l$, we will be able to now extend the integral relative to $r$ beyond this limit, and if we want to be until $r = \infty$. If we put also

$$\frac{2\pi}{3}\int_0^{\infty} Vr^2\ dr \equiv k, \tag{16}$$

where, $k$ will be a function of $u,\ x,\ y,\ z,$ and we will have

$$\frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dx}\ r^2\ dr = \frac{dk}{dx},\quad \frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dy}\ r^2\ dr = \frac{dk}{dy},\quad \frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dz}\ r^2\ dr = \frac{dk}{dz}\ ;$$

in consequence, the general equation of the motion of the heat will come to be finally

$$(7)_{PS4}\qquad c\frac{du}{dt} = \frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz}. \tag{17}$$

When all the point of $A$ gets to a stationary state, we will have $\frac{du}{dt} = 0$, and then it will result

$$\frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz} = 0,$$

for the equation relative to this stationary state.

[19]

## §50.

The equation (17) coincides with that which I found in years ago for the case of a heterogeneous corps [20], however, in never supposing hence that the quantity $k$ depended

[18](⇓) According to [56, p.41, no.277],

$$\int \cos^m x\ \sin x dx = -\frac{\cos^{m+1} x}{m+1}.$$

[19](⇓) The expression (15) is reduced into

$$c\frac{du}{dt} = \Big(\frac{d^2u}{dx^2}k + \frac{du}{dx}\frac{dk}{dx}\Big) + \Big(\frac{d^2u}{dy^2}k + \frac{du}{dy}\frac{dk}{dy}\Big) + \Big(\frac{d^2u}{dz^2}k + \frac{du}{dz}\frac{dk}{dz}\Big) \tag{18}$$

[20]sic. *Journal de l'École Polytechnique*, 19$^e$ cahier, page 87. (⇓) Poisson [64], [76, p. 677].

on the temperature $u$.

If $A$ is a corps heterogeneous,

- $k$ will depend only on $u$,
- and the equation (17) will be changed as follows :

$$(8)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{dk}{du}\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}\right). \tag{19}$$

[21] In supposing that this quantity $k$ were independent of $u$, we could have the equation

$$(9)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \tag{20}$$

[22] which we give it ordinarily, and which is reduced, in the case of the stationary state, to an equation independent of two quantities $c$ and $k$, viz.,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0. \tag{21}$$

[23]

After obtained the equation (20), in considering $c$ and $k$ as the constant quantities, we could suppose

- that it will conserve the same form when these quantities will variable,
- that it will suffice to put here for $\frac{k}{c}$ a function given with $u$,
- and that the equation relative to the stationary state doesn't receive any change.

However, it is seen that these suppositions are never admissible ; the equation (20) and here one which is deduced in the case of $\frac{du}{dt} = 0$, were never, in the same case of a homogeneous corps, the exact equation of the motion of the heat and that of the stationary state ; and the formula (19) shows that the independence of partial differences [a] of $u$ of the second order in respect to $x,\ y,\ z$, the true equations need also to contain the square of its partial differences [b] of the primary order.

[a](⇓) id.
[b](⇓) id.

To have regard to displacement of points of $A$, products with the dilations and condensations due to variation of the temperature, or from another cause, we will replace, as we mentioned above, $\frac{du}{dt}$ with the formula (7), and the equation (17) will come to be

$$(10)_{PS4} \qquad c\left(\frac{du}{dt} + \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} + \frac{du}{dz}\frac{dz}{dt}\right) = \frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz}. \tag{22}$$

[21](⇓) Because of $k = k(u)$, from each second terms in the right hand-side of the expression (18) is reduced into

$$\left(\frac{du}{dx}\frac{dk}{dx}\right) + \left(\frac{du}{dy}\frac{dk}{dy}\right) + \left(\frac{du}{dz}\frac{dk}{dz}\right) = \left(\frac{du}{dx}\frac{du}{dx}\frac{dk}{du}\right) + \left(\frac{du}{dy}\frac{du}{dy}\frac{dk}{du}\right) + \left(\frac{du}{dz}\frac{du}{dz}\frac{dk}{du}\right) = \frac{dk}{du}\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}\right)$$

[22](⇓) The equation (20) means $c\frac{du}{dt} = k\Delta u$, where $\Delta$ meaning the Laplacian.

[23](⇓) This function $u$ satisfying the equation (21) is called harmonic function. Poisson doesn't mention the harmonic function, however, Poincaré [78, p.237] calls it so. cf. Table. 1.

Here is this equation (22) which we will come to joint, for instance, to the ordinary equations of the motion of liquids, to accomplish it, hence that I proposed already in my *Study of Mechanics*[24] and in a preceding memoir. [25]

Poisson puts also the another heat equations [26] and in a preceding memoir. [27]

## 7. Part 2. The derivative productions of classical heat analyses

### 7.1. Detail items. the derivative productions of classical heat analyses.

1. We discuss historical development of the particular value in the wave analysis, including Prévost 1792 [81], Physico-Mechanical Researches of the Heat, Fourier 1822 [22], Analytic Theory of the Heat, and Poisson 1835 [77], Mathematical Theory of the Heat and finally Poincaré 1895 [78] Analytic Theory of Propagation of Heat. 2. In this 18-19 century, the conception of continuum is introduced at first by Laplace, many mathematician challenge the physico-mathematical problems. One in Prévost's essay on heat is the communication theory of heat, which becomes Fourier's main and initial motif in his scholar life. 3. After Laplace, Fourier and Navier, et al. participate in these studies, and Fourier puts forth the trigonometric series in the process of building the heat theory, including communication theory and the theory of heat motion in fluid. 4. In the rivalry with Fourier, Poisson puts forth his personality independent of Fourier, the digressions on the mathematics : these are his characteristic, namely, on the mathematical analysis of the integral, the partial equations, and the trigonometric series. Poisson traces many historical facts of the origins of the wave equations including the trigonometric series by the trailblazers such as Euler, Lagrange, Laplace, Fourier, etc. 5. Poincaré puts forth many conceptions of pure analysis to solve the flux of heat from the viewpoint of up-to-date mathematical physics such as theory of Dirichret, theorem of Abel, theorem of Cauchy, theory of asymptotic value, theory of singular points, theory of holomorphic function, meromorphic function, etc. 6. We talk about the derivative productions of classical heat analyses such as particular value and eigenvalue, trigonometric series and its convergence, linear integral equation, meromorphic function, terrestrial system, or meteorology, etc. from the widely comparative viewpoint in the history of mathematics or mathematical physics.

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[24] (⇓) *Traité de Mécanique*, op. cit. cf. Poisson [59], [75] and [76].

[25] (⇓) Poisson puts also the another heat equations such as in Chapter 6. entitled : Digression on the integral of the partial differential equations. §76. [77, p.146], or Chapter 11. entitled : Distribution of the heat in certain corps, and specially in a homogeneous sphere primitively heated with a certain manner. §162. [77, p.347] :

$$(1)_{PS11} \qquad \frac{du}{dt} = a^2\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right), \qquad \frac{k}{c}=a^2, \quad \Rightarrow \quad \frac{du}{dt}=a^2\Delta u. \tag{23}$$

where, $u$ is the heat, $k$ and $c$ are the conductibility and the specific heat of the material. $\Delta$ is the Laplacian.

[26] (⇓) *Traité de Mécanique*, op. cit. cf. Poisson [75, 76],

[27] (⇓) Poisson puts also the another heat equations such as in Chapter 6. entitled : Digression on the integral of the partial differential equations. §76. [77, p.146], or Chapter 11. entitled : Distribution of the heat in certain corps, and specially in a homogeneous sphere primitively heated with a certain manner. §162. [77, p.347] :

$$(1)_{PS11} \qquad \frac{du}{dt} = a^2\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right), \qquad \frac{k}{c}=a^2, \tag{24}$$

where, $u$ is the heat, $k$ and $c$ are the conductibility and the specific heat of the material.

TABLE 3. The five books and one paper on physico-mathematical theories of heat

| | Name | Prévost<br>1792 [81]<br>(1751-1839) | Laplace<br>1818<br>(1749-1827) | Fourier<br>1822 [22]<br>(1768-1830) | Poisson<br>1835 [77]<br>(1781-1840) | Dirichlet<br>1837 [14]<br>(1806-59) | Poincaré<br>1895 [78]<br>(1854-1912) |
|---|---|---|---|---|---|---|---|
| 1 | title | *Recherches physico-mécaniques sur la chaleur* | *Traité de mécanique céleste* | *Théorie analytique de la chaleur* | *Théorie mathé-matique de la chaleur* | *Über die Darstellung ganz willkürlicher Functionen durch Sinus- und Cosinus-reihen* | *Théorie analytique de la propagation de la chaleur* |
| 2 | total page | 232 | | [22] 541<br>except for contents<br>[23] 641 | [77] 552 | [14] 26 | 314 |
| 3 | point/ merit | communi-cation theory<br>· only quantity of heat depends on the heat communi-cation, not with temperature | many mathematical concepts | · heat theory<br>· trigonometric series | · check on Fourier's method mathemati-cally with another method<br>· proving convergence of series | · theorem of Dirichlet<br>· proof of convergence of the series | introduction of<br>· Dirichlet principle,<br>· harmonic function<br>· the methods by Fourier, Laplace, Cauchy, Riemann<br>· analogous equations with heat such as the equation of cord vibration, telegraph |
| 4 | contri-butions | to Fourier's communi-cation theory | continuum theory to the hydrodynamics & heat dynamics like Fourier, Navier, Poisson, Poincaré | · trigonometric series after followers until now<br>· molecular theory to Navier | to Sturm and Liouville | to Poincaré<br>· Dirichlet principle | introduction of various conceptions like :<br>· integral equ.<br>· harmonic func.<br>· holomorphic func.<br>· meromorphic func.<br>· spherical func.<br>· spherical trigonometry<br>· spherical polynomial |
| 5 | other relative papers | | · *Mémoire du flux et du reflux*, 1790<br>· *Connaisances des Tems*, 1823 | 1805 [18],<br>1808 [10],<br>1816 [19],<br>1824 [24],<br>1826 [25],<br>1827 [26],<br>1829 [27],<br>1835 [28],<br>1890 [11], | 1808 [58],<br>1823 [67],<br>1823 [68],<br>1823 [69],<br>1824 [70] | 1829 [12]<br>1830 [13] | |
| 6 | remark | Poisson [77] introduces [81] as an essay | | | Poincaré 1895 doesn't mention this at all | check on Fourier's proving | Fredholm [29] refers on Poincaré's harmonic function |

TABLE 4. The function, theory, law and introduction of preceding work of heat

| | Name | Prévost<br>1792 [81]<br>(1751-1839) | Laplace<br>1818<br>(1749-1827) | Fourier<br>1822 [22]<br>(1768-1830) | Poisson<br>1835 [77]<br>(1781-1840) | Dirichlet<br>1837 [14]<br>(1806-59) | Poincaré<br>1895 [78]<br>(1854-1912) |
|---|---|---|---|---|---|---|---|
| 1 | title | *Recherches physico-mécaniques sur la chaleur* | *Traité de mécanique céleste* | *Théorie analytique de la chaleur* | *Théorie mathématique de la chaleur* | *Über die Darstellung ganz willkürlicher Functionen durch Sinus- und Cosinus-reihen* | *Théorie analytique de la propagation de la chaleur* |
| 2 | Theory/<br>theorem | | | Lagrange<br>Laplace | Lagrange<br>Laplace<br>Fourier | | theorem of Fourier<br>theprem of Cauchy<br>Bessel<br>theorem of Dirichlet<br>Dirichlet condition<br>theorem of Abel |
| 3 | law(l)/<br>formula/<br>notation(n) | | Laplacian,<br>Laplace equ. | Lagrange<br>Laplace | Taylor<br>Lagrange<br>Laplace<br>Fourier<br>Poisson equ.<br>Poisson brachet<br>(n)Legendre | | (l)Newton<br>Taylor<br>Laplace<br>Fourier<br>Green<br>(n)Halphen |
| 4 | introduction of preceding work of heat | | | Biot<br>Laplace<br>Poisson | Biot<br>Jakob Bernoulli<br>Prévost 1792 [81]<br>Laplace<br>Fourier | | Laplace<br>Fourier<br>Cauchy<br>Abel |
| 5 | math-ematical consider-ation | Heat transfer independly of the temperature | | trigono-metric series | three digressions<br>· ¶71-91.<br>· ¶92-104.<br>· ¶105-115.<br>cf. Remark in the out of Table ??. | 1829 [12]<br>1830 [13]<br>1837 [14] | ¶43-4 theorem of Abel and its application<br>¶ 57 Integral of Fourier<br>¶ 62 $\int_0^\infty \varphi_1(y) \frac{\sin \alpha y}{y} dy$<br>¶ 127 uniqueness of development<br>¶ 136 $\iint RU_i d\tau = 0, \quad \forall i \leq n$<br>¶ 157 condition of Dirichlet |
| 6 | numerical calculation or experiment | | | | inequality of temperature in day/year/place | | |
| 7 | newness | Quantity of heat is only concerned in the transfer of heat. Temperature isn't concern-ed in it. | · continuum<br>· molecular action | · trigono-metric series<br>· heat equation<br>· heat diffusion equation | · molecular radiation<br>· terrestrial heat including<br>sterate heat<br>atmospheric heat<br>solar hear<br>· mereorology | proving of Fourier's unproved | · electric wave<br>· telegraph equ.<br>· pure mathematics such as<br>· holomorphy,<br>· meromorphy,<br>· etc. |

## 8. The origin of eigenvalue problem

Euler 1748 [16] says the height of the vibrating cord is calculated by the linear, first-ordered expression as follows :

$$y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \gamma \sin \frac{3\pi x}{a} + \cdots \tag{25}$$

Lagrange 1759 [44] descrives as the introductional expression of the trigonometric series by $P_\nu$ and $Q_\nu$ as follows :

$$P_\nu \equiv Y_1 \sin \frac{\nu \varpi}{2m} + Y_2 \sin \frac{2\nu \varpi}{2m} + Y_3 \sin \frac{3\nu \varpi}{2m} + \cdots + Y_{m-1} \sin \frac{(m-1)\nu \varpi}{2m} \tag{26}$$

where, $Q_\nu$ has the same linear, first-ordered combination with coefficients $V_1,\ V_2,\ \cdots$ instead of $Y_1,\ Y_2,\ \cdots$. The indicies of $P$ and $Q$ show simply the *valeur particulières* (eigenvalues) of $\nu$ which (the *valeur particulières*) belong to them ($P_\nu$ and $Q_\nu$, respectively). [44, pp.79-80] (trans. mine.) Remark. Lagrange's $\varpi$ is equal to $\pi$. In (25), we can see in case we assume $a = 2m$ and $\alpha,\ \beta,\ \gamma,\ \cdots$ are equal to $Y_1,\ Y_2,\ Y_3,\ \cdots$, then $x = \nu$ in Lagrange's $P_\nu$ in (26) or $Q_\nu$ reffering to *valeur particulière* (eigenvalue), namely (25) = (26).

## 9. The derivative productions of classical heat analyses

### 9.1. *La valeur particulière* and the eigenvalue.

We confirm the identity of *valeur particulière* with the eigenvalue. We would pay attention to the historical fact that it has been developed for the linear differential equation on the heat deffusion, or the trigonometric serirs and eigenvalue problem in the analysis including string or sonic oscillation, the rapidly changing (decreasing/increasing) function,[28] and the process redefined of the eigenvalue by Hilbert in 1904.

- We think the eigenvalue is translated from *la valeur particulière* into German word *der Eigenwert* by the Hilbert 1904 and is expatiated by Courant-Hilbert 1924 [9]. The word *eigenfunction* is combined corresponding to the word : *eigenvalue.*
- In the bibliographies of the earlier centuries, for example, Lagrange 1759, Fourier 1822, Poisson 1823, 1835, Cauchy 1823, Sturm 1836, Liouville 1836, Poincaré 1895, et al. use la *valeur particulière.* Sturm and Liouville owe to Poisson's preceding works of now so-called Sturm-Liouville type differential equation of the second order.
- In the first English translation of Fourier's main work [22], Freeman 1878 [30] uses 'the particular value' to all the over 43 original words in this book.
- Wilkinson 1952 uses *eigenvalue* without using the other English word : *proper value* or particular value in recognition of its nomenclature of eigenvalue.
- Today's French word : *la valeur propre,* used by Chatelin 1988, et al., may be reimported from German *Eigenwert* after Wilkinson's English word *eigenvalue.*
- The then French usage of *la fonction particulière* / *le espace particulière* corresponding to the eigenfunction / eigenspace / eigenvector aren't distinct in these days, however, the correspondency between the eigenvalue and the function is visible, for example, such as the expression in Poisson [77] or Sturm [89] or the

[28] cf. We cite the rapidly changing function in §48 of 6.1.

expression in Liouville [47], in spite of the fact that its usage aren't so distinct as after Hilbert.

- On the other hand, the word *valeur caractèristique* aren't used as the eigenvalue.
- At last, we can recognize Euler 1748 on the cord vibration as one of the origin of eigenvalue problem. It is because the two equations (25) and (26) are the same.

## 10. The carried-over to the next century unifying the legacies in the 19th C.

In 1878, ten years earlier than G. Darboux, A. Freeman [30] published the first English translated Fourier's second version 1822. To this work, Lord Kelvin (William Thomson) contributes to import the Fourier's theory into the England academic society. [29] The microscopical description of hydromechanics equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations. After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872. They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1905 , in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose from Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845. From Fourier's equation of heat, Boltzmann's gas transport equation is deduced. (cf. Table 1, 2).

## 11. Poisson's contributions

Poisson contributes in making his paradigm to the fluid dynamics and heat theory are as follows :

- He presents the 'two constant theory', which we assert,[30] as visible in the Navier-Stokes equations in 1831. After this, Stokes follows Poisson's equation in 1849, and Prandtl declears these equations as the 'Navier-Stokes equations' in the top of the twenty century.
- He proposes the alternative method of the definite integral, [31] instead of making the universal method of it, since by Euler, Lagrange and Laplace.
- He evaluates the trigonometric series by Lagrange as the original and analytical series and which is enhanced and succeeded to the Fourier's series.
- He shows the heat equation by deducing precisely, although Fourier's series is the first, however, its introduction isn't deducing such as Poisson's or without demonstration.
- Although his approach dues to the rivarly to the Fourier's theory, it brings up the derivative productions of the another solutions or thinking in making many breakthroughs to Fourier's method.

## 12. General Conclusions

1. We consider our problem as the totality among the definite integral, the trigonometric series, etc., for Poisson's objection to Fourier is relating the universal and

[29] A.Freeman puts the name of W. Thomson in his acknoledgement. cf. [30, errata].
[30] (⇓) cf. Section 10, and Table 1, 2.
[31] (⇓) cf. [60], [76, pp.347-368] and [77, pp.129-182].

TABLE 5. The family of eigenvalue/eigenfunction. Rem. (·) : frequency used. $n+$ : more than $n$. B2 : fonction harmonique by Poincaré. e.v.: eigenvalue, e.f.: eigenfunction, e.s.: eigenspace, Ew : Eigenwert, Ef : Eigenfunction, Er : Eigenraum, numberless A and B : using implicitly.

| no | English | (A)proper value (B)proper function/ space | particular value | indole value indole function | e.v. e.f./e.s. | remark |
|---|---|---|---|---|---|---|
| | French | valeur propre, function propre | valeur particulière, function particulière /espace particulière | valeur characteristique function charactèristique | | (B2) fonction harmonique |
| | Latin | | | value indole function indole | | |
| | German | | | | Ew,Ef/Er | |
| 1 | Lagrange | | 1760-61, [39] (A:1+) | | | |
| 2 | Laplace | | 1782, 85 [40] (A:5+) | | | |
| 3 | Lacroix | | 1800 [38] (A:4) | | | |
| 4 | Fourier | | 1822 [22] (A:43+, B) | | | |
| 5 | Poisson | | 1808 [58] (A:3), 1823 [68] (A:18), 1831 [74] (A:7+), 1835 [77] (A:45, B:1) | | | |
| 6 | Cauchy | | 1815 [5] (A:1) 1823 [6] (A:5) 1823 [7] (A:6) | | | |
| 7 | Gauss | | | 1830 [31] cf. [48] | | |
| 8 | Sturm | | 1836 [88] (A:6), 1836 [89] (A:3, B) | | | |
| 9 | Liouville | | 1836 [47] (A, B) | | | |
| 10 | Freeman (translation of Fourier[22]) | | 1878 [30] (A:43+) (translation of Fourier [22]) | | | |
| 11 | Poincaré | | 1895 [78] (A:1) | | | 1895 [78] (B2:5) |
| 12 | Hilbert | | | | 1904 [35], [36] | |
| 13 | Courant-Hilbert | | | | 1924 [9] | |
| 14 | Schrödinger | | | | 1926 [85, 86] | |
| 15 | Gerschgolin | | | | 1931 [32] | |
| 16 | Wilkinson | | | | 1952 [91] | |
| 17 | Chatelin | 1988 [8] | | | | |

fundamental problem of analytics, as we show Poisson's analytical/mathematical thought or sight in the Chapter 6, etc. In fact, Poisson's work-span covers them.

2. Fourier doesn't show the pricise deduction of the heat equation (5), while Poisson takes 9 pages to descrive it from §44 to §50. The diffefrence between Fourier

and Poisson is the common kernel function of molecular distance, which Poisson manipulates in both fluid motion and heat motion.

3. Boltzmann's concept of collision and transport with entropy and probability are treated as the classical quantum mechanics. In this sense, Fourier's communication theory and the equation of motion in the fluid stand on the communication point between the classical mechanics and new quantum mechanics by Schrödinger.
4. Owing to the arrival of continuum, we are able to discuss the solution of the problem on the continuous space of mathematics. As Duhamel [15] says, at first, Poisson performs it with the concept of mathematically infinite continuity. This allows us to discuss, without depending on the microscopic-description, by the vectorially description, like Saint-Venant, Stokes.
5. Although the confusion of knowledges on continuum, the unity in the mathematics are gained, however, the applicabilities of the unite or general equations are then not yet defined, which comes from the misunderstandings interphysico-mathematics, such as the identity of fluid and elasticity, or, fluid and heat.
6. Sturm-Liouville type differential equations of heat diffusion problems [47, 88] are redefined by Hilbert [36] using the second order differential operator $\mathcal{L}$ and as the *EigenWert* problem translating from the traditionally used nomenclature *la valeur particulière.*
7. About the describability of the trigonometric series of an arbitrary function, nobody succeeds in it including Fourier, himself. Up to the middle of or after the 20th century, these collaborations are continued, finally in 1966, by Carleson proved in $L^2$, and in 1968, by Hunt in $L^p$.

## 13. Epilogue

Poisson [77, pp.411-415] expects the earth warming before the Industorial Revolution [32] up to 17 years after. According to his speculation, in using this average rate of the increment per a year is $0.22°C$, then we can estimate with this increment rate up to this year 2015, just at the COP21, the temperature rises between 198 years, $2.447°C$ as follows :

$$\frac{11.950 - 11.730}{17\frac{7}{12}} \times 198 = \frac{0.22}{17.58333} \times 198 = 2.477°C.$$

This is what is called the reason of the consensus about the increment of the earth warming in the world.

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