A.H. Clifford (1908-1992) の「群の表現論」に関する業績

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References

【添付文書】 論文 [Hir] の最終稿コピー（plus Appendices）

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◆ **Alfred Hoblitzelle Clifford** (July 11, 1908 – December 27, 1992)
【注. William Kingdon Clifford (4 May 1845 – 3 March 1879)】

H. Weyl の助手 (1936–1938) (from a biographical article [B1]):
In 1933 Clifford was awarded his doctorate for his dissertation entitled *Arithmetic of Ova.* (中略) After the award of his doctorate, Clifford became a member of the Institute for Advanced Study in Princeton. He remained there for five years and during this period, from 1936 to 1938, he was Weyl's assistant. This was the time when Weyl was writing *The classical groups* and he wrote in the Preface:

> If at least the worst blunders of expression have been avoided, this relative accomplishment is to be ascribed solely to the devoted collaboration of my assistant, Dr Alfred H. Clifford and even more valuable to me than the linguistic, were his mathematical criticism.

Weyl's influence is clearly seen in Clifford's papers in 1937 *Representations induced in an invariant subgroup.* (論文 [4], [5]) In these he considered the representation induced on a normal subgroup by an irreducible representation of the group. (後略)

Clifford の仕事 [4], [5] に注目した動機.

我々は現在次のような問題に取り組んでいる.『ある種の群の増大列 $\{G_n\}_{n\geqslant 1}$ とその帰納極限 $G_\infty := \lim_{n\to\infty} G_n$ を考える：

$$G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots \subset G_\infty.$$

ここで，G_n の双対 $\widehat{G_n}$ を考える．G_n の既約表現 π_n の指標 χ_{π_n} は同値類 $[\pi_n] \in \widehat{G_n}$ を代表する．既約指標の列 $\{\chi_{\pi_n}\ ;\ [\pi_n] \in \widehat{G_n},\ n = 1, 2, \ldots\}$ に対して $n \to \infty$ のときの漸近挙動や極限 $\lim_{n\to\infty} \chi_{\pi_n}$ を考察して，群 G_∞ の表現や指標との関連を論ずる．さらに，G_n の線形表現の代わりに射影表現を考えて同種の問題を論ずる.』

我々が Clifford の仕事 [4], [5] に注目した主たる動機は，次の （1），（2） である：

（1） 各 G_n のすべての既約射影表現を構成する方法を与えること.

（2）（線形表現以外に）射影表現をも重要視する根拠を与えること.

1933 [1] A system arising from a weakened set of group postulates

[1] A system arising from a weakened set of group postulates, Ann. Math., **34**(1933), 865-871.

【注． この論文は，後日の semigroup の研究への最初の一歩であろう.】

群を定義する公理系を考察する．集合 G に対して，

I. G には２項演算がある： $G \times G \ni (a, b) \mapsto ab = c \in G$.

II. 結合律が成立する： $(ab)c = a(bc)$.

III. 少なくとも１つの左単位元がある： $\exists e \in G$ s.t. $\forall a \in G,\ ea = e$.

逆元の存在に関しては，次の４種類の公理が考え得る：

IV L. (左逆元の存在) $\forall a \in G$, $\forall$ 左単位元 e, に対して，**少なくとも１つ**の左逆元 b が存在する： $ba = e$.

IV R. (右逆元の存在) $\forall a \in G$, $\forall$ 左単位元 e, に対して，**少なくとも１つ**の右逆元 b が存在する： $ab = e$.

V L. (左逆元の '存在') $\forall a \in G$, **少なくとも１つ**の左単位元 e, に対して，**少なくとも１つ**の左逆元 b が存在する： $ba = e$.

V R. (右逆元の '存在') $\forall a \in G$, **少なくとも１つ**の左単位元 e, に対して，**少なくとも１つ**の右逆元 b が存在する： $ab = e$.

◆ 公理系 (I, II, III, IV L) は群を定義する.

定理 1. 公理系として，(I, II, III, IV R), (I, II, III, V L), (I, II, III, V R) は互いに同値である.

証明. (I, II, III, IV R) $\Longrightarrow$ (I, II, III, V R) なので，

(i) (I, II, III, V R) $\Longrightarrow$ (I, II, III, V L),
(ii) (I, II, III, V L) $\Longrightarrow$ (I, II, III, IV R),
を示せば，ぐるっと一回りして全部が同値になる．

(i) の証明． $\forall a \in G$, **少なくとも 1 つの**左単位元 e, に対して，**少なくとも 1 つの**右逆元 b が存在： $ab = e$. そこで，$ba =: c$ とおくと，
$baba = c^2$, $bea = c^2$, $\therefore ba = c^2$, $\therefore c = c^2$.
d を c の 1 つの右逆元とすると，$cd = e'$ (e' は 1 つの左単位元),
$cd = c^2 d \Longrightarrow e' = ce'$.
$\forall x \in G$ に対して，$cx = ce'x = e'x = x$, ゆえに c は左単位元．よって，$ba = c$ は (I, II, III, V L) を意味する． 【(i) 証了】

(ii) の証明． $\forall a \in G$, **少なくとも 1 つの**左単位元 e', に対して，**少なくとも 1 つの**左逆元 b が存在する： $ba = e'$.
c を b の左逆元とする： $cb = e''$. すると，$bab = e'b = b$, $\therefore cbab = cb$, $\therefore e''ab = e''$, $\therefore ab = e''$. かくて，$\forall$ 左単位元 e に対し，
$a(be) = e''e = e$, 従って，a の右逆元は be である【(ii) 証了】 □

弱い公理系 (I, II, III, IV R) を満たす G を **multiple group** という．

◆ multiple group が右単位元を持てば群である．

定理 2. multiple group G においては， $ca = cb \Longrightarrow a = b$.

定理 3. multiple group G においては， $ab = e$ (左単位元) $\Longrightarrow ba = e'$ (左単位元).

定理 4. multiple group G においては， $\forall a \in G$, $\forall e$ 左単位元，に対して，$\exists_1$ a の右逆元．a の勝手な左逆元 b をとる（$ba = e$）と，be がそれである： $a(be) = e$.

証明． $ab = e'$ より，$abe = e'e = e$ □

定理 5. multiple group G においては， $\forall a \in G$ に対して，$\exists_1 e$ 左単位元 s.t. a は e に対して左逆元を持つ．さらに，$ae = ea = a$ であり，他の左単位元 $e' \neq e$ に対しては a と e' とは可換ではない．

証明． $ba = e$, $b'a = e'$ とすると，$abe = e$, $ab'e' = e'$.
??? □

$\alpha := |\{$ 左単位元 $\}|$ を G の**次数** (index) とよぶ．

◆ $\alpha = 1$ ならば，G は群である．

$\{e_i \,;\, i \in I\}$ を左単元の全体とする ($\alpha = |I|$).

e_i に関する a の右逆元（定理 4）を $\underline{a}_i^{-1}$ と書く： $a\underline{a}_i^{-1} = e_i$.
a が左逆元を持つ左単位元を e_j とする（一意的，定理 5）.
定理 3, 4 により，$\underline{a}_i^{-1} a = e_j$ ($\forall i \in I$).
また，定理 4 により，$\underline{a}_i^{-1} e_k = \underline{a}_k^{-1}$ ($k \in I$).

$K_i := \{x \in G \,;\, x$ は e_i に対して，左逆元を持つ $\}$: $G = \bigsqcup_{\alpha \in I} K_i$.

定理 6. multiple group G においては，各 K_i は群をなす．
（これらを G を構成する群（groups of composition）という.）
さてそこで，$a_i \in K_i$ をとり，$a_i e_j$ を調べよう．$b := a_i e_j$ とおく．a_i^{-1} を群 K_i における逆元とすると，$a_i^{-1} a_i = e_i$. すると，$a_i^{-1} b = e_i e_j = e_j$,
$b = a_i e_j$ は e_j に対して左逆元を持つので，$b \in K_j$. そして，
$be_i = a_i e_j e_i = a_i e_i = a_i, \quad b = a_i e_j =: a_j \ (j \neq i), \quad \begin{cases} a_i e_j = a_j, \\ a_j e_i = a_i, \end{cases}$
元 $\{a_i\,;\, i \in I\}$ を互いに **共役**（conjugate）という．

定理 7. 共役対応 $a_i \leftrightarrow a_j$ により K_i, K_j は互いに同型である： $K_i \cong K_j$.
（multiple group G を構成する群 K_i 達と同型な群を **composition group** とよぶ.）

定理 8. multiple group G を共役類に分けるとその共役類の集合は composition group と同型な群になる．

定理 9. 左単位元ばかりからなる multiple group を **unitary multiple group** というが，それはその次数によって一意的に決まる： $e_i e_j = e_j \ (i, j \in I)$. multiple group G の unitary multiple subgroup は G の次数 α で決まる．

定理 10. H を任意の群，α を任意の基数とする．このとき，H を composition group とし，α を次数とする multiple group G を構成できる．
逆に，任意の multiple group G はその composition group H と次数 α とで決まる．

証明. $|I| = \alpha$ となる添字集合をとる．$i \in I$ に対し，H のコピー K_i をとり，$a \in H$ に対応する元を $a_i \in K_i$ とする．$G := \bigsqcup_{i \in I} K_i$ とし，積の定義を

$$a_i b_j := (ab)_j \quad (a, b \in H,\ i, j \in I)$$

とする．これは公理系 (I, II, III, V L) を満たす．従って，G は multiple group である．

さて今度は，G を multiple group とする．定理 7 におけるように $K_i \ni a_i \leftrightarrow a_j \in K_j$ とすると，$a_i e_j = a_j$ なので，
$a_i b_j = a_j b_j$. 他方，$a_j b_j = (ab)_j$, ゆえに，$a_i b_j = (ab)_j$. □

定理 11. multiple group G において，$a, b \in G$ に対し，方程式 $ax = b$ は一意的な解を持つ．

例 1. $G = \boldsymbol{C}^\times$, $a, b \in G$ の積は， $(a, b) := |a|b$.
公理系 (I, II, III, IV R) が成立する．左単位元の集合は $\{e^{i\varphi}\,;\, \varphi \in \boldsymbol{R}\}$.
composition group は $\boldsymbol{R}_+^\times$.

1937 [5] Representations induced in an invariant subgroup

[5] Representations induced in an invariant subgroup, Ann. Math., **38**(1937), 533-550.

【注．群 G の既約線形表現を G の正規部分群に制限したときに，その構造を見ると，自然に射影表現が現れていること，を示した後半部分が重要である．
原論文の記号を適当に変更するなどして分かり易くして，かなり書き換えてある．】

1 表現の制限 $\pi|_N$ の完全可約性

G： 抽象群
N： G の正規部分群
π： G の体 P 上の行列表現
　$r \in G$ の π への作用： $({}^r\pi)(s) := \pi(r^{-1}sr)$ $(s \in G)$.

定理 1.1. π を G の体 P 上の行列による既約表現，N を正規部分群とする．このとき，π の制限 $\pi|_N$ はそれ自身既約であるか，もしくは，同一次元の N の既約表現の直和に分解する．$\rho_N^{(1)}$ をその既約成分の 1 つとすると，他の既約成分は G のもとで，これに共役である．さらに，これに G-共役な任意の既約表現は $\pi|_N$ に現れる．

証明のキーポイント． π の表現空間を V とし，その N-不変部分空間を V' とすると，「任意の $r \in G$ に対して，$\pi(r)V'$ も N-不変である．」実際，$u \in N$, $v' \in V'$ に対し，

$$\pi(u)\big(\pi(r)v'\big) = \pi(r)\big(\pi(r^{-1}ur)v'\big) \in \pi(r)V' \quad (\because\ u' = r^{-1}ur \in N).$$

2 表現 π の準原始性（imprimitivity）

定義 2.1. G の表現 π が**準原始的**（または**半原始的**，**非原始的**）であるとは，次の条件を満たすときである：
π の表現空間 $V(\pi)$ は直和

$$V(\pi) = V_1 + V_2 + \cdots + V_k \tag{1}$$

に分解して，$\{V_i\,;\,1 \le i \le k\}$ は各 $\pi(g)$ $(g \in G)$ によって，全体として置換される．

定理 1.1 の状況において，$\rho_N^{(1)}, \rho_N^{(2)}, \ldots, \rho_N^{(m)}$ を $\pi|_N$ に現れる既約成分のうちで，互いに非同値なものの最大の集合とする．i $(1 \le i \le m)$, に対して，V_i を $\rho_N^{(i)}$ と同値な既約表現の働く部分空間の和，とする．

定理 2.1. $V_1, V_2, \ldots, V_m$ は π の半原始性の系をなす．$\pi|_N$ の各既約成分 $\rho_N^{(i)}$ は同一の重複度 l で $\pi|_N$ に現れ，$\dim\pi = lmn$, $n = \dim\rho_N^{(i)}$.

証明． 分解 $V(\pi) = V_1 + V_2 + \cdots + V_m$ に従って $\pi(g)$ の行列表示を整理して

$$\pi(g) = \begin{pmatrix} \pi_{11}(g) & \cdots & \pi_{1m}(g) \\ \cdots & \cdots & \cdots \\ \pi_{m1}(g) & \cdots & \pi_{mm}(g) \end{pmatrix}. \tag{2}$$

とする．$u \in N$ にたいしては，π_{ii} は $\rho_N^{(i)}$ の l 重の直和で，$\pi_{ij}(u) = 0$ $(i \neq j)$ である．

$g \in G$ に対して，

$$\pi(g)^{-1}\pi(u)\pi(g) = \pi(g^{-1}ug) = ({}^g\pi)(u) \quad (u \in N),$$
$$\pi_{ii}(u)\,\pi_{ij}(g) = \pi_{ij}(g)\pi_{jj}(g^{-1}ug) \quad (u \in N). \tag{3}$$

であるから，${}^g(\rho_N^{(j)})(u) = \rho_N^{(j)}(g^{-1}ug) \cong \rho_N^{(i)}(u)$ $(u \in N)$ でなければ，$\pi_{ij}(g) = 0$．こうした j は一意的なので，$j = g^{-1}(i)$ とおく．

これにより，$\pi(g)$ は $V_{g^{-1}(i)} \to V_i$ $(1 \leq i \leq m)$ と $V_j, j = g^{-1}(i)$, を V_i に移して置換する．π が既約なので，この置換は遷移的であり $\dim V_i$ は同一である．定理 1.1 により，$\dim \rho_N^{(i)} = n$ $(\forall i)$, 従って，$\rho_N^{(i)}$ の $\pi|_N$ における重複度は同一でなければならぬ． □

注 2.1. 上の定理により，$\pi(g)$ のブロック型表示 (2) において，各行各列にはそれぞれ 1 個だけ零でない $\pi_{ij}(g)$ がある $(j = g^{-1}(i),\ j = g(i))$． □

注 2.2 (imprimitive の訳語). 術語 *im*primitive は，primitive（原始的）の否定だから，その訳語として，「非原始的」が有り得るが，この語は「原始的」の否定 (not primitive) を表す．しかし，用語「imprimitive」で表されている上の状況は「primitive からは少し外れているが primitive に近い」もしくは「primitive でありたいが，今少し足りない」といった状況である．従って，全否定ではない訳語「準原始的」もしくは「半原始的」の方が状況をより適切に表していると思われる．ここでは「準原始的」を採用した． □

$$H := \{g' \in G\ ;\ \pi(g')V_1 = V_1\}, \tag{4}$$

とおく．集合 $\{r_1 = e, r_2, r_3, \ldots, r_m\} \subset G$ を $\pi(r_i)V_1 = V_i$ となるように選ぶと，

$$G = \bigsqcup\nolimits_{1 \leqslant i \leqslant m} r_i H. \tag{5}$$

π の行列表示 (2) の第 1 列に注目すると，$\pi_{i1}(r_i) :\ V_1 \to V_i$ で $\pi_{i1}(r_i)$ は正則，また，$\pi_{1i}(r_i^{-1}) :\ V_i \to V_1$, も正則である．

$$L := \begin{pmatrix} E & & & & \\ & \pi_{21}(r_2) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \pi_{m1}(r_m) \end{pmatrix} \quad (E\ \text{は単位行列}), \tag{6}$$

とおき（L は固定行列），$\pi(g)$ を $L^{-1}\pi(g)L$ と変換して，これをあらためて $\pi(g)$ と書く．

$\pi_{i1}(r_i) = E$ で，$\pi_{1i}(r_i^{-1}) = E$ となり，$\pi(r_i h r_j^{-1}) : V_j \to V_1 \to V_1 \to V_i$ で，

$$\pi_{ij}(r_i h r_j^{-1}) = \sum_{k,k'} \pi_{ik}(r_i)\pi_{kk'}(h)\pi_{k'j}(r_j^{-1}) = \pi_{11}(h) \quad (h \in H), \tag{7}$$

なぜなら，$k = 1$, $k' = 1$ が $\pi_{ik}(r_i)$ と $\pi_{k'j}(r_j^{-1})$ が零でない唯一の添字だから．

$\tau_H(h) := \pi_{11}(h)$ とおく.

ここであらためて，任意の $g \in G$ をとる．勝手な j に対し，$\pi(g) : V_j = \pi(r_j)V_1 \to V_i = \pi(r_i)V_1$ $(i = g(j),\ 1 \leq j \leq m)$ である．言い換えると，

$$r_i^{-1} g r_j \in H$$

となる i が一意に存在する．そして，

(8)
$$\pi_{ij}(g) = \begin{cases} \tau_H(r_i^{-1} g r_j), & r_i^{-1} g r_j \in H \text{ のとき}, \\ 0, & \text{その他のとき}. \end{cases}$$

H の表現 τ_H は既約でなければならぬ．そして，G の表現 π は H の既約表現 τ_H から作られた半原始的表現である．$\tau_H|_N = [l]\cdot\rho_N^{(1)}$ であるが，この τ_H は次小節で調べる.

注 2.3. Frobenius が有限群の表現論を創成した時期の初期の論文 [F56] (1898) での「部分群からの誘導表現」の理論は主として「指標の誘導」について書いてあるが，これを「表現の誘導」に書き換えて（有限群からすこし拡張して）使えば，$\pi_{ij}(g)$ の公式 (8) は，『$\pi(g) = \big(\pi_{ij}(g)\big)$ は，τ_H の誘導表現である』を意味するのが分かる：

(9)
$$\pi \cong \mathrm{Ind}_H^G \tau_H\,.$$

3 表現 τ_H の構造 （原論文の §4 の内容も取り込んで書き直してある）

この小節では基礎体 P は代数的閉であると仮定する．$\tau := \tau_H$, $\rho := \rho_N^{(1)}$ とおく.

定理 3.1. H を群，N をその正規部分群，τ を H の既約表現とする．制限 $\tau|_N$ の既約成分はある ρ に同値である，と仮定する：

(10)
$$\tau|_N \cong [l]\cdot\rho\,.$$

(i) H の既約表現 τ は H の 2 つの既約射影表現の直積に同値である：

(11)
$$\tau(h) = C(h) \times \Gamma(h).$$

ここに，$\dim C = \dim\rho$, $\rho(h^{-1}uh) = C(h)^{-1}\rho(u)C(h)$ $(h \in H, u \in N)$, $\dim\Gamma = l$.

(ii) 次が成立するように正規化できる：$h \in H$, $u \in N$ に対して，

$$C(hu) = C(h)\rho(u),\ C(u) = \rho(u);\quad \Gamma(hu) = \Gamma(h),\ \Gamma(u) = E_l.$$

このとき，$h \mapsto C(h)$ は H の射影表現であり，$h \mapsto \Gamma(h)$ は 実質上 商群 H/N の射影表現である．それらに付随する因子団は実質上 $H/N \times H/N$ 上の関数であり，互いに他の逆である.

証明. τ の表現空間を適当に分解して，$\tau(u)$ $(u \in N)$ の行列表示が次の対角型になるようにする：

(12)
$$\tau(u) = \begin{pmatrix} \rho(u) & & & & \\ & \rho(u) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \rho(u) \end{pmatrix} \quad (u \in N),$$

これに対応して $\tau(h)$ $(h \in H)$ をブロック型に表示する：

$$
(13) \qquad \tau(h) = \begin{pmatrix} \tau_{11}(h) & \cdots & \cdots & \cdots & \tau_{1l}(h) \\ \tau_{21}(h) & \cdots & \cdots & \cdots & \tau_{2l}(h) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{l1}(h) & \cdots & \cdots & \cdots & \rho_{ll}(h) \end{pmatrix} \quad (h \in H).
$$

仮定により，$\forall h \in H$ に対して，$n \times n$ 行列 $C(h)$ $(n = \dim \rho)$ が存在して，

$$
(14) \qquad \rho(h^{-1}uh) = C(h)^{-1}\rho(u)C(h) \quad (u \in N).
$$

$C(h)$ はスカラー倍を除いて決まるので，

$$
h \mapsto C(h) \quad (h \in H)
$$

は，H の射影表現であり，

$$
(15) \qquad C(hh') = \alpha(h, h')C(h)C(h') \quad (h, h' \in H,\ \alpha(h, h') \in \boldsymbol{C}^{\times})
$$

ここに，$\alpha(h, h')$ は射影表現 C の因子団である．$\tau(h)^{-1}\tau(u)\tau(h) = \tau(h^{-1}uh)$, から，

$$
\begin{aligned}
&\rho(u)\tau_{\alpha\beta}(h) = \tau_{\alpha\beta}(h)\rho(h^{-1}uh), \\
(16) \qquad \therefore \quad &\rho(u)\bigl(\tau_{\alpha\beta}(h)C(h)^{-1}\bigr) = \bigl(\tau_{\alpha\beta}(h)C(h)^{-1}\bigr)\rho(u),
\end{aligned}
$$

となる．P が代数的閉なので，Schur の補題が使えて，

$$
\tau_{\alpha\beta}(h) = \gamma_{\alpha\beta}(h)C(h) \quad (\exists \gamma_{\alpha\beta}(h) \in \boldsymbol{C}^{\times}),
$$

となる．$l \times l$ 行列 $\Gamma(h) = \bigl(\gamma_{\alpha\beta}(h)\bigr)$ を使えば，

$$
(17) \qquad \tau(h) = C(h) \times \Gamma(h) \quad (h \in H).
$$

$\tau(hh') = \tau(h)\tau(h')$ $(h, h' \in H)$ なので，(15) から，

$$
(18) \qquad \Gamma(hh') = \alpha(h, h')^{-1}\Gamma(h)\Gamma(h'),
$$

を得て，Γ も H の射影表現であること，および，その因子団が C のそれの逆であることが分かる．$\tau(u) = \rho(u) \times E_l$ なので，$C(u) = \rho(u)$ $(u \in N)$ と仮定できる．すると，

$$
(19) \qquad \begin{cases} \Gamma(hu) = \alpha(h, u)\Gamma(h) \\ \Gamma(h) = \alpha(h, u)^{-1}\Gamma(hu) \end{cases} \quad (h \in H, u \in N).
$$

ここで，因子団 $\alpha(h, h')$ が実質上 $H/N \times H/N$ 上の関数であることを示そう．$C(h)$ に対する (14) において，h を hv $(v \in N)$ に置き換えると，

$$
\rho(h^{-1}uh) = \bigl(\rho(v)C(hv)^{-1}\bigr)\rho(u)\bigl(\rho(v)C(hv)^{-1}\bigr)^{-1}.
$$

故に，再び Schur の補題により，$C(hv) = \lambda_{h,v}\, C(h)\rho(v)$ $(\exists \lambda_{h,v} \in \boldsymbol{C}^{\times},\ h \in H,\ v \in N)$.

そこで，H/N の完全代表系 $\mathcal{R}$ を勝手に取り，新しく，

(20) $$C(h_0u) := C(h_0)\rho(u), \quad C(u) = \rho(u) \quad (h_0 \in \mathcal{R},\ u \in N),$$

とおく（古い $C(h)$ とはスカラー倍しか違わない）．すると，

(21) $$C(hu) = C(h)\rho(u)\ (h \in H, u \in N)$$

となる．よって，$u, v \in N,\ h, k \in H$ に対して，

$$\begin{aligned} C(hu)C(kv) &= C(h)\rho(u)C(k)\rho(v) \\ &= C(h)C(k)\,\rho(k^{-1}uk)\rho(v) \\ &= \alpha(h,k)C(hk)\rho(k^{-1}ukv) = \alpha(h,k)C(hukv), \\ \therefore \quad & \alpha(hu, kv) = \alpha(h,k). \end{aligned}$$

定理 3.1 の (11) を使うと，$h \in H, u \in N$ に対して，

$$\begin{aligned} \tau(hu) &= C(hu) \times \Gamma(hu), \\ \tau(hu) &= \tau(h)\tau(u) = \big(C(h) \times \Gamma(h)\big)\big(\rho(u) \times E_l\big) \\ &= \big(C(h)\rho(u)\big) \times \Gamma(h) = C(hu) \times \Gamma(h), \\ \therefore \quad & \Gamma(hu) = \Gamma(h) \quad (h \in H, u \in N). \\ \therefore \quad & \Gamma(hu)\Gamma(kv) = \alpha(k,h)\Gamma(hk). \end{aligned}$$

このとき，$h \to \Gamma(h)$ は H/N の射影表現で，その因子団 $\alpha(h,k)$ $(h,k \in H)$ は $H/N \times H/N$ 上の関数である． □

4 埋め込み問題

問題 4.1. 正規部分群 N の既約表現 ρ は，G のある既約表現に埋め込まれるか？

G が有限群，もしくは，コンパクト群であれば，答は YES.

二つの条件を考える：

(4.2.1) $H := \{s \in G\ ;\ {}^s\rho \cong \rho\}$ とするとき，$[G : H] < \infty$,

(4.2.2) ρ は H のある既約表現 τ に含まれる．

定理 4.1. **(i)** N の既約表現 ρ が G のある既約表現 π に埋め込まれるための必要十分条件は，(4.2.1)+(4.2.2) である．

(∵) 誘導表現 $\tau = \mathrm{Ind}_N^H\rho$ を考えよ．

(ii) 基礎体 P を代数的閉とする．(4.2.2) が成立するための必要十分条件は

(4.2.2′) $({}^h\rho)(u) = C(h)\rho(u)C(h)^{-1}$ $(u \in N,\ h \in H)$ により決まる H の射影表現 C を (21) が成立するように正規化する．その因子団 $\alpha(h,k)$ $(h,k \in H)$ は $H/N \times H/N$ 上の関数になる．因子団 $\alpha(h,k)^{-1}$ を持つ H/N の有限次元射影表現 Γ が存在する．

このとき，$\tau = C \times \Gamma$ とおくと，$\tau(u) = \rho(u) \times \Gamma(e) = \rho \times E_l$.

(iii) もし $[G : N] < \infty$ ならば，任意の N の既約表現は G のある既約表現に含まれる． (∵) 誘導表現 $\pi = \mathrm{Ind}_N^G\rho$ を考えよ．

5 随伴関係

ここでは，N の既約表現 ρ が埋め込まれる G の既約表現 π をすべて求める方法を調べる.

定義 5.1. G の 2 つの既約表現 π_1, π_2 が N に関して**随伴**しているとは，$\pi_1|_N, \pi_2|_N$ が共通の既約成分を持つことである.

定義 5.2. H の 2 つの表現 Γ, Δ が **同値** (strictly equivalent) であるとは，定行列 M があって，次が成立すること：

$$\Delta(h) = M^{-1}\Gamma(h)M \quad (h \in H).$$

定理 5.1. N の既約表現 ρ に関して随伴関係にある G の 2 つの既約表現 π_1, π_2 は，それらが決める H/N の既約射影表現 Γ_1, Γ_2 の分だけ違い得る：$\pi_1 \cong \pi_2 \iff \Gamma_1 \cong \Gamma_2$.

6 半線形（semi-linear）表現への拡張

群 G は基礎体 P にも作用しているとする：$G \times P \ni (s, \alpha) \mapsto {}^s\alpha \in P$. すると，自然とベクトル空間 $V = P^n$ にも働く.

G の**半線形**表現 $s \mapsto \{A(s), s\}$ とは，$s \in G$ に対し，写像

$$V \ni v \mapsto A(s)({}^s v) \in V,$$

を与えるものである．この範疇においても，上と同様の結果が得られる．（以下略）

2012 半直積群への，[5] 定理 3.1 の応用（平井）

【注. 論文 [5] 定理 3.1 から，この応用に至るまでには，かなりの距離があった筈だが，一旦，応用としての（以下の）定理 1.1（既約表現の構成法）が証明されてみると，それほどでもないとも思える．なお，後日に，この定理 1.1 の（[5] とは独立の）別証明（Mackey の理論とも独立）を得て，[Hir, 2013] に書いた.】

$G = N \rtimes S$ を正規部分群 N と部分群 S との半直積群とする．$s \in S$ は N 上に働き，従ってその表現 ρ にも，$({}^s\rho)(u) := \rho(sus^{-1})$ $(u \in N)$ と働く.

1 半直積群 G の既約表現の構成法

さらに S を有限と仮定する．このとき，G の任意の既約表現は次の手順で得られる：

(1) N の既約表現 ρ をとり，その同値類 $[\rho]$ の固定化部分群を $S([\rho])$ と書く：

$$S([\rho]) := \{s \in S\ ;\ {}^s\rho \cong \rho\}, \tag{1}$$

そして $H(\rho) := N \rtimes S([\rho])$ とおく.

(2) $s \in S([\rho])$ に対して，${}^s\rho$ と ρ との相関作用素 $C(s)$ を具体的に求める：

$$C(s)^{-1}\rho(u)C(s) = {}^s\rho(u) = \rho(s^{-1}us) \quad (u \in N). \tag{2}$$

すると $C(s)$ はスカラー倍を除いて確定するので，$S([\rho]) \ni s \to C(s)$ は射影表現である．その因子団 α は，

$$C(s)C(t) = \alpha(s,t)C(st) \quad \big(s,t \in S([\rho])\big), \tag{3}$$

である． $C(us) := \rho(u)C(s) \quad \big(s \in S([\rho]),\ u \in N\big)$, とおく．これは半直積群 $H(\rho)$ の射影表現で，その因子団は，つぎを満たす：

$$\alpha(us, vt) := \alpha(s,t) \quad \big(s,t \in S([\rho]),\ u,v \in N\big).$$

(3) $S([\rho])$ の因子団 $\alpha(s,t)^{-1}$ $\big(s,t \in S([\rho]\big)$ を持つ既約射影表現 κ をとる．そして，

$$\tau(us) := \big(\rho(u)C(s)\big) \boxtimes \kappa(s) = C(us) \boxtimes \kappa(s) \ \ \big(u \in N,\ s \in S([\rho])\big), \tag{4}$$

とおくと，τ は，2つの因子団が互いに消しあって，$H(\rho)$ の通常の線形表現になる．

そこで，下の様に誘導表現をとれば，これが G の既約表現を与える：

$$\pi[\rho,\kappa] := \mathrm{Ind}_{H(\rho)}^{G}\tau\,. \tag{5}$$

定理 1.1. **(i)** 半直積群 $G = N \rtimes S$ において，N をコンパクト群，S を有限群と仮定する．このとき，G の任意の既約表現 π は (4) のどれかの $\pi[\rho,\kappa]$ に同値である．

(ii) $\pi[\rho,\kappa] \cong \pi[\rho',\kappa'] \iff \rho \cong \rho'$ かつ $\kappa \cong \kappa'$.

証明. Clifford [5] の定理 3.1 や定理 5.1 を応用する．しかし，それらから上の定理 1.1 に直ちに至るというわけではなく，何ほどかの距離がある．そこで，この定理 1.1 の証明は読者におまかせするので，試みられよ． □

注. 定理 1.1 の別証明を [Hir] で与えた．これは，Clifford [5] や Mackey の仕事からは独立である．

2 射影表現が現れる半直積群の例 $GL(2^k, \boldsymbol{C}) \rtimes S$

2.1. 特別の次元 $n = 2^k$ の一般線形群を $N = GL(2^k, \boldsymbol{C})$ とおき，これを正規部分群とする半直積群 $G = N \rtimes S$ を考える．N に作用する群 S を次の様に取る：

$$S = \begin{cases} \mathfrak{S}_{2k+1}, & \text{in Case I}\,; \\ \mathfrak{S}_{2k+1}, & \text{in Case II}_-\,; \\ \mathfrak{A}_{2k+1}, & \text{in Case II}\,; \\ \mathfrak{S}_{2k}, & \text{in Case II}', \end{cases} \tag{6}$$

それぞれの場合に，複素代数 $\mathcal{A} := \mathfrak{M}(2^k, \boldsymbol{C}) \supset N = GL(2^k, \boldsymbol{C}) = \mathcal{A}^\times$ への S の作用を与えるために，まずこの代数の特殊な生成元系を与える．そのためにまず，2 次のユニタリ行列を Schur に従って，

$$\varepsilon = \begin{pmatrix}1&0\\0&1\end{pmatrix},\ a = \begin{pmatrix}0&1\\1&0\end{pmatrix},\ b = \begin{pmatrix}0&-i\\i&0\end{pmatrix},\ c = \begin{pmatrix}1&0\\0&-1\end{pmatrix},\ (i = \sqrt{-1}) \tag{7}$$

とおく．この行列 a, b, c は後年 Pauli が水素原子などの電子の記述に関して発見した Pauli の 3 行列 $\sigma_x, \sigma_y, \sigma_z$ と同じものである．これらは次の関係式をみたす: $[a,b] := ab - ba$, とおいて,

$$(8)\qquad [a,b] = 2ic,\ [b,c] = 2ia,\ [c,a] = 2ib,\ abc = i\varepsilon.$$

さらに，次数 $n = 2^k$ の行列 Y_j $(j \in \boldsymbol{I}_{2k+1} := \{1, 2, \ldots, 2k+1\})$ を

$$(9)\qquad \begin{cases} Y_{2i-1} = c^{\otimes(i-1)} \otimes a \otimes \varepsilon^{\otimes(k-i)} & (i \in \boldsymbol{I}_k), \\ Y_{2i} \quad = c^{\otimes(i-1)} \otimes b \otimes \varepsilon^{\otimes(k-i)} & (i \in \boldsymbol{I}_k), \\ Y_{2k+1} = c^{\otimes(k-1)} \otimes c, \end{cases}$$

とおく．ここに，$x^{\otimes p}$ は x の p 回テンソル積を表すが，$p = 0$ のときには無視する．

補題 2.1. $E = E_{2^k}$ を 2^k 次の単位行列とする．$\mathcal{G}_k := \{Y_j\ (j \in \boldsymbol{I}_{2k+1})\}$ は代数 $\mathfrak{M}(2^k, \boldsymbol{C})$ の生成元系であり，その基本関係式は，次で与えられる：

$$(10)\qquad \begin{cases} {Y_j}^2 \ = \ E & (j \in \boldsymbol{I}_{2k+1}), \\ Y_j Y_l \ = \ -Y_l Y_j & (j, l \in \boldsymbol{I}_{2k+1},\ j \neq l), \end{cases}$$

$$(11)\qquad Y_1 Y_2 \cdots Y_{2k+1} = i^k E.$$

定義 2.1. (6) 式で与えられた群 S に対し，$\sigma \in S$ の系 $\{\pm Y\,;\, Y \in \mathcal{G}_k\}$ への作用を次のようにおく：

$$(12)\qquad \begin{cases} \text{Case I}: & \sigma^{\mathrm{I}}(Y'_j) := \mathrm{sgn}(\sigma)\, Y'_{\sigma(j)} \quad (j \in \boldsymbol{I}_{2k+1}); \\ \text{Case II}_-: & \sigma^{\mathrm{II}_-}(Y_j) := \mathrm{sgn}(\sigma)\, Y_{\sigma(j)} \quad (j \in \boldsymbol{I}_{2k+1}); \\ \text{Case II}: & \sigma^{\mathrm{II}}(Y_j) := Y_{\sigma(j)} \quad (j \in \boldsymbol{I}_{2k+1}); \\ \text{Case II}': & \sigma^{\mathrm{II}'}(Y_j) := \begin{cases} Y_{\sigma(j)} & (j \in \boldsymbol{I}_{2k}) \\ \mathrm{sgn}(\sigma)\, Y_{2k+1} & (j = 2k+1), \end{cases} \end{cases}$$

ここに，$Y'_j := (-1)^{j-1} Y_j$ $(j \in \boldsymbol{I}_{2k+1})$.

定理 2.2. (i) 定義 2.1 の σ^{I} (または σ^{II_-}) は，$\mathfrak{S}_{2k+1}$ の複素代数 $\mathcal{A} = \mathfrak{M}(2^k, \boldsymbol{C})$ への作用を与える．

(ii) σ^{II} は，群 $\mathfrak{A}_{2k+1}$ の複素代数 $\mathcal{A}$ への作用を与える．

(iii) $\sigma^{\mathrm{II}'}$ は $\mathfrak{S}_{2k}$ の $\mathcal{A}$ への作用を与える．

(iv) これらの作用は，群 $G = GL(2^k, \boldsymbol{C}) = \mathcal{A}^\times$ への作用を誘導する．
（それらは同じ記号で表される）．

2.2. 第 (6) 式の群 S と $N = GL(2^k, \boldsymbol{C})$ との半直積の群

$$(13)\qquad G = N \rtimes S$$

の既約表現の構成を与えるのに必要な基本的事実を与えよう．$\sigma \in S$ の $X \in \mathfrak{M}(2^k, \boldsymbol{C})$ への作用を（ここでだけ）統一的に $X \mapsto {}^\sigma X$ と表す．「複素代数 $\mathfrak{M}(2^k, \boldsymbol{C})$ の同型はす

べて内部自己同型である」という性質により，同型 ${}^{\sigma}X$ は，すべてある行列 $\nabla(\sigma)$ による共役作用で表される：

$$(14) \qquad {}^{\sigma}X = \nabla(\sigma)X\nabla(\sigma)^{-1} \quad \big(X \in \mathfrak{M}(2^k, \boldsymbol{C})\big).$$

この線形写像 $\nabla(\sigma)$ を作用 $X \to {}^{\sigma}X$ の**相関作用素**（intertwining operator）とよぶ．$\nabla(\sigma)$ はスカラー倍を除いて決まるので，対応 $S \ni \sigma \mapsto \nabla(\sigma)$ は多価性を持ち，したがって，S の射影表現である可能性がある．

定理 2.3. 群 $\mathfrak{S}_n$ の単純互換 $s_p = (p\ p{+}1)$ による生成元系 $\{s_p\,;\, p \in \boldsymbol{I}_{n-1}\}$【または $\mathfrak{A}_n$ の生成元系 $\{s_q s_p\,;\, q > p \geq 1\}$】に対して，その相関作用素は，それぞれの場合ごとに次のように与えられる：

$$(15) \qquad \begin{cases} \nabla_n(s_p) &:= \frac{(-1)^{p-1}}{\sqrt{2}}(Y_p + Y_{p+1}) = \frac{1}{\sqrt{2}}(Y'_p - Y'_{p+1}), \\ \nabla'_n(s_p) &:= \frac{1}{\sqrt{2}}(Y_p - Y_{p+1}), \\ \nabla''_n(s_p) &:= \nabla'_n(s_p)\cdot iY_{2n'+1} \qquad (i = \sqrt{-1}). \\ \mho_n(s_q s_p) &:= \nabla'_n(s_q)\nabla'_n(s_p) = \nabla''_n(s_q)\nabla''_n(s_p) \quad (q > p \geq 1). \end{cases}$$

証明. 計算によって求められる．ただし，相関作用素の具体形が与えられた段階では，相関関係 (14) を $X = Y_j$（生成元）に対してチェックするだけでよいが，それらは次の補題に列挙されている． □

補題 2.4. (i) $n = 2k$ または $n = 2k+1$，$\sigma \in \mathfrak{S}_n$ に対して，

$$(16) \qquad \begin{cases} \nabla_n(s_p)Y'_j\nabla_n(s_p)^{-1} = -Y'_{s_p(j)} & (p \in \boldsymbol{I}_{n-1},\ j \in \boldsymbol{I}_n), \\ \nabla_n(\sigma)X\nabla_n(\sigma)^{-1} = \sigma^{\mathrm{I}}(X) & \big(X \in \mathfrak{M}(2^k, \boldsymbol{C})\big)\,; \end{cases}$$

$$(17) \qquad \begin{cases} \nabla'_n(s_p)Y_j\nabla'_n(s_p)^{-1} = -Y_{s_p(j)} & (p \in \boldsymbol{I}_{n-1},\ j \in \boldsymbol{I}_n), \\ \nabla'_n(\sigma)X\nabla'_n(\sigma)^{-1} = \sigma^{\mathrm{II}-}(X) & \big(X \in \mathfrak{M}(2^k, \boldsymbol{C})\big), \end{cases}$$

(ii) $n = 2k$，$\sigma \in \mathfrak{S}_{2k}$ に対して，

$$(18) \qquad \begin{cases} \nabla''_n(s_p)Y_j\nabla''_n(s_p)^{-1} = Y_{s_p(j)} & (p \in \boldsymbol{I}_{n-1},\ j \in \boldsymbol{I}_n), \\ \nabla''_n(\sigma)X\nabla''_n(\sigma)^{-1} = \sigma^{\mathrm{II}'}(X) & \big(X \in \mathfrak{M}(2^k, \boldsymbol{C})\big). \end{cases}$$

(iii) $n = 2k+1$, $\widetilde{\mathfrak{A}}_{2k+1}$, に対して，

$$(19) \qquad \begin{cases} \mho_n(v_i)\,Y_j\,\mho_n(v_i)^{-1} = Y_{s_{i+1}s_1(j)} \\ \qquad\qquad (v_i = s_{i+1}s_1,\ i \in \boldsymbol{I}_{2k-1},\ j \in \boldsymbol{I}_{2k+1}), \\ \mho_n(\sigma)\,X\,\mho_n(\sigma)^{-1} = \sigma^{\mathrm{II}}(X)\ \big(= \sigma^{\mathrm{II}-}(X)\big) \\ \qquad\qquad \big(\sigma \in \mathfrak{A}_{2k+1},\ X \in \mathfrak{M}(2^k, \boldsymbol{C})\big). \end{cases}$$

定理 2.5.

(i) ∇_n $(n = 2k)$ と ∇'_n $(n = 2k+1)$ は群 $\mathfrak{S}_n$ の射影表現である．
(ii) $n = 2k$ に対して，∇''_n は群 $\mathfrak{S}_n$ の射影表現である．
(iii) $n = 2k+1$ に対し，$\mho_n$ は群 $\mathfrak{A}_n$ の（２価の）射影表現である．

1940 [7] Partially ordered abelian groups

[7] Partially ordered abelian groups, Ann. Math., **41**(1940), 465-473.

（省略）

1941 [8] Factor sets of a group in its abstract unit group

[8] (with Saunders MacLane) Factor sets of a group in its abstract unit group, Trans. Amer. Math. Soc., **50**(1941), 385-406.

1 論文内容の概要

定義 1. 有限群 Γ に対して，その**抽象単位群**（abstract unit group）$\mathfrak{H}$ とは，生成元系 $\{H^\sigma\,;\,\sigma\in\Gamma\}$ と基本関係式

(1) $$\prod\nolimits_{\sigma\in\Gamma} H^\sigma = 1,$$

で定義される群である．この群の同型として，$\tau\in\Gamma$ に対して，

(2) $$A(\tau):\ H^\sigma \mapsto H^{\sigma\tau} \quad (\sigma\in\Gamma),$$

とおくと，$A(\tau)A(\tau') = A(\tau\tau')$ $(\tau,\tau'\in\Gamma)$ である．

定義 2. $\mathfrak{H}$ の Γ による**拡張**（extension）$\mathfrak{G}$ とは，$\mathfrak{G}$ が $\mathfrak{H}$ を正規部分群として含み，$\mathfrak{G}/\mathfrak{H}\cong\Gamma$ となるものである．

$\mathfrak{G}$ 内の $\mathfrak{G}/\mathfrak{H}$ の完全代表元系 u_σ $(\sigma\in\Gamma)$ をとって，$\mathfrak{G}$ の元を $u_\sigma A$ $(\sigma\in\Gamma,\ A\in\mathfrak{H})$ と表すと，その積演算は $\sigma,\tau\in\Gamma$, $A\in\mathfrak{H}$ に対し，

(3) $$Au_\sigma = u_\sigma A^\sigma, \quad u_\sigma u_\tau = u_\sigma F_{\sigma,\tau}\ (F_{\sigma,\tau}\in\mathfrak{H}),$$

と書ける．$\Gamma\times\Gamma$ 上の $\mathfrak{H}$-値関数として，$F_{\sigma,\tau}$ は

(4) $$F_{\rho,\sigma}{}^{\tau} F_{\rho\sigma,\tau} = F_{\rho,\sigma\tau} F_{\sigma,\tau}$$

をみたす．これを Γ の $\mathfrak{H}$ における**因子団**（factor set）とよぶ．

別の完全代表元系 $u'_\sigma = u_\sigma C_\sigma$ $(\sigma\in\Gamma,\ C_\sigma\in\mathfrak{H})$ をとると，これに対する因子団は，

(5) $$F'_{\sigma,\tau} = \frac{C_\sigma{}^{\tau} C_\tau}{C_{\sigma\tau}} F_{\sigma,\tau} \quad (\sigma,\tau\in\Gamma)$$

である．因子団 $F_{\sigma,\tau}$ と $F'_{\sigma,\tau}$ とは，**同伴**であるという．同伴な因子団を持つ，$\mathfrak{H}$ の 2 つの拡張 $\mathfrak{G}$, $\mathfrak{G}'$ も**同伴**であるという．

定義 3. $F\mathfrak{H}$:= すべての $\mathfrak{H}$-値因子団のなす可換群，
$T\mathfrak{H}$:= $\mathfrak{H}$-値因子団で自明な因子団 **1** に同伴なものからなる可換群，
$M\mathfrak{H}$:= $F\mathfrak{H}/T\mathfrak{H}$ を Γ の $\mathfrak{H}$ における**乗法因子団**（mutiplicator）とよぶ．

定義 4. Ω を代数的閉体で，その標数が $n=|\Gamma|$ を割らないものとする．$\Gamma\times\Gamma$ 上の Ω-値関数 $\omega_{\sigma,\tau}$ が，Ω における Γ の**因子団**とは，

$$\omega_{\rho,\sigma}\omega_{\rho\sigma,\tau}=\omega_{\rho,\sigma\tau}\omega_{\sigma,\tau}\quad(\rho,\sigma,\tau\in\Gamma), \tag{6}$$

を満たすものである．Γ の Ω における**乗法因子団** $\mathfrak{M}$ も同様に定義する．

予想 1. Γ の $\mathfrak{H}$ における乗法因子団 $M\mathfrak{H}$ は $\mathfrak{M}$ と同型である．

解答. Γ が可解群ならば OK

$$V\mathfrak{H}:=\{C=(C_\sigma)_{\sigma\in\Gamma}\ ;\ C_\sigma\in\mathfrak{H}\}\cong\mathfrak{H}^n,$$

$$\varphi:\ V\mathfrak{H}\ni C=(C_\sigma)_{\sigma\in\Gamma}\longmapsto F=(F_{\sigma,\tau})_{\sigma,\tau\in\Gamma}\in T\mathfrak{H},\ \ F_{\sigma,\tau}:=\frac{C_\sigma^\tau C_\tau}{C_{\sigma\tau}}, \tag{7}$$

$$U\mathfrak{H}:=\{C=(C_\sigma)_{\sigma\in\Gamma}\in V\mathfrak{H}\ ;\ C_\sigma^\tau C_\tau=C_{\sigma\tau}\}=\mathrm{Ker}(\varphi),$$

$$V\mathfrak{H}/U\mathfrak{H}\cong T\mathfrak{H}.$$

$U\mathfrak{H}$ の元 $C=(C_\sigma)$, $C_{\sigma\tau}=C_\sigma^\tau C_\tau$，を**交差指標**（crossed character）という．（**注意.** semilinear map と類似している．）

$\psi:\ \mathfrak{H}\ni D\longmapsto\psi(D)=(C_\sigma),\ C_\sigma:=D^{1-\sigma}:=\dfrac{D}{D^\sigma}\in\psi(\mathfrak{H})$

$C=\psi(D)$ は交差指標である．実際，

$C_{\sigma\tau}=\dfrac{D}{D^{\sigma\tau}}=\dfrac{D^\tau}{(D^\sigma)^\tau}\dfrac{D}{D^\tau}=C_\sigma^\tau C_\tau.$

定義 5. $C\mathfrak{H}:=U\mathfrak{H}/\psi(\mathfrak{H})$ を Γ の**交差指標群**とよぶ．

予想 2. Γ の交差指標群 $C\mathfrak{H}$ は Γ の線形指標群 $\mathfrak{C}$ と同型である．

解答. Γ が可解群ならば OK

（以下，論文の詳細は割愛する．）

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付録

Classical method of constructing all irreducible representations of semidirect product of a compact group with a finite group[1]

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Abstract. Let $G = U \rtimes S$ be a group of semidirect product type, with U compact and S finite. For an irreducible representation (=IR) ρ of U, let $S([\rho])$ be the stationary subgroup in S of the equivalence class $[\rho] \in \widehat{U}$. Intertwining operators $J_\rho(s)$ $\big(s \in S([\rho])\big)$ between ρ and ${}^{s^{-1}}\rho$ gives in general a spin (= projective) representation of $S([\rho])$, which is lifted up to a linear representation J'_ρ of a covering group $S([\rho])'$ of $S([\rho])$. Put $\pi^0 := \rho \cdot J'_\rho$, and take a spin representation π^1 of $S([\rho])$ corresponding to the factor set inverse to that of J_ρ, and put $\Pi(\pi^0, \pi^1) = \mathrm{Ind}^G_{U \rtimes S([\rho])}(\pi^0 \boxdot \pi^1)$. We give a simple proof for that $\Pi(\pi^0, \pi^1)$ is irreducible and that any IR of G is equivalent to some of $\Pi(\pi^0, \pi^1)$.

Introduction. Let $G = U \rtimes S$ be a semidirect product group, with U compact and S finite. For an irreducible representation (=IR) ρ of U, let $S([\rho])$ be the stationary subgroup in S of the equivalence class $[\rho] \in \widehat{U}$. Intertwining operators $J_\rho(s)$ $\big(s \in S([\rho])\big)$, defined by $\rho\big(s(u)\big) = J_\rho(s)\, \rho(u)\, J_\rho(s)^{-1}$ $(u \in U)$, gives in general a spin (= projective) representation of $S([\rho])$, which is lifted up to a linear representation J'_ρ of a certain covering group $S([\rho])'$ of $S([\rho])$ (cf. Lemma 1.2). Put $\pi^0 := \rho \cdot J'_\rho$. Take a spin representation π^1 of $S([\rho])$ corresponding to the factor set inverse to that of J_ρ, then the tensor product $\pi^0 \boxdot \pi^1$ is a non-spin IR of $U \rtimes S([\rho])$. Inducing $\pi^0 \boxdot \pi^1$ up to G, we get $\Pi(\pi^0, \pi^1) = \mathrm{Ind}^G_{U \rtimes S([\rho])}(\pi^0 \boxdot \pi^1)$. We give a simple proof for the following theorem (cf. Theorem 4.1).

Theorem. *Each $\Pi(\pi^0, \pi^1)$ is irreducible and the set of $\Pi(\pi^0, \pi^1)$ given above is complete in the sense that the dual $\widehat{G}$ of G has a complete set of representatives in it, or that any irreducible representation of G is equivalent to some of $\Pi(\pi^0, \pi^1)$'s.*

Our proof is elementary and needs only the least minimum on projective representations of groups, prepared in §1. (See [Schl] or [Yam] for more knowledges on general theory of such representations.) For proofs of irreducibility and completeness, we utilize fully characters of induced representations. Thus, our proof is completely independent of the results of A.H. Clifford in [Clif], with which we can give another proof.

This paper is organized as follows. In §1, we give some preparatory lemmas on projective representations of groups in relation to their central extensions. In §2, we treat the case of semidirect product groups $G = U \rtimes S$ with U compact and S finite, and give the definition of $\Pi(\pi^0, \pi^1) = \mathrm{Ind}^G_{U \rtimes S([\rho])}(\pi^0 \boxdot \pi^1)$ with $\pi^0 = \rho \cdot J'_\rho$. In §3, an integral expression of the character of $\Pi(\pi^0, \pi^1)$ is given, and using it the irreducibility of $\Pi(\pi^0, \pi^1)$ is proved. Also we give a set $\Omega(G)$ of $\Pi_{i,j} = \Pi(\pi^0_i, \pi^1_j)$, whose characters $\chi_{\Pi_{i,j}}$ form an orthonormal system in the subspace of invariant functions in $L^2(G)$. In §4, the main theorem, Theorem 4.1, is established by proving the completeness of the set $\Omega(G)$ of characters $\chi_{\Pi_{i,j}}$, first in the case of finite groups, then in the case of U compact in general. In the latter case, we use a fundamental lemma, Lemma 4.2.

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[1] 本編は，Probability and Mathematical Statistics, **33**(2013), 掲載論文の投稿直前版に少々の改変を加え，２つの Appendixes を付加したものである．

1 Preparatory lemmas

1.1. Central extension. Let G' be a topological group and Z its closed central subgroup. Put $G := G'/Z$, then the following sequence is exact:

$$(8) \qquad 1 \longrightarrow Z \longrightarrow G' \longrightarrow G \longrightarrow 1 \quad \text{(exact)},$$

and G' is called a *central extension* of G by Z. We call G' also as a covering group of G, and call G as the base group of G'.

A representation Π of G' is called *of spin type* $\chi \in \widehat{Z}$ if $\Pi(z) = \chi(z)I$ $(z \in Z)$, where I denotes the identity operator. Denote by $\widehat{G'}$ the dual of G' consisting of equivalence classes $[\Pi]$ of unitary irreducible representations (=IRs) Π of G', and by $\widehat{G'}^{\chi}$ its subset consisting of $[\Pi]$ such that the spin type of Π is $\chi \in \widehat{Z}$. For a compact group H, denote by μ_H the normalized Haar measure on H such that $\mu_H(H) = 1$, and denote by $L^2(H)$ the Hilbert space of L^2-functions on H with respect to μ_H.

A function f on G' is called *of spin type* χ if $f(zg') = \chi(z)f(g')$ $(z \in Z, g' \in G')$. Assume G' is compact. For $\chi \in \widehat{Z}$, denote by $L^2(G';\chi)$ the subspace of $L^2(G')$ consisting of all $f \in L^2(G')$ of spin type χ. Then $L^2(G')$ is an orthogonal direct sum of $L^2(G';\chi)$ over $\chi \in \widehat{Z}$. A matrix element f of a representation Π of spin type χ is a spin function of the same type. Take a complete set of representatives $\Omega^{\chi} = \{\Pi\}$ of $\widehat{G'}^{\chi}$, and denote by $\mathcal{M}([\Pi])$ the space spanned by matrix elements of Π, then $L^2(G';\chi)$ is an orthogonal direct sum of $\mathcal{M}([\Pi])$ over $\Pi \in \Omega^{\chi}$.

Lemma 1.1. (i) *Let G' be finite, then for each $\chi \in \widehat{Z}$,*

$$\sum_{[\Pi]\in\widehat{G'}^{\chi}} (\dim \Pi)^2 = \frac{1}{|Z|}\,|G'| = |G|.$$

(ii) *Let G' be infinite compact, then for each $\chi \in \widehat{Z}$, the number of equivalence classes in $\widehat{G'}^{\chi}$ is infinite.*

Proof. (i) The dimensions of $\ell^2(G';\chi)$ is equal to the sum of $\dim \mathcal{M}([\Pi])$ over $[\Pi] \in \widehat{G'}^{\chi}$. Matrix elements of Π span $\mathcal{M}([\Pi])$, and $\dim \mathcal{M}([\Pi]) = (\dim \Pi)^2$. This gives us $\dim \ell^2(G';\chi) = \sum_{[\Pi]\in\Omega^{\chi}} (\dim \Pi)^2$. On the other hand, a function f on G' belongs to $\ell^2(G';\chi)$ if and only if it satisfies $f(zg') = \chi(z)f(g')$ $(z \in Z,\ g' \in G')$. Hence $\dim \ell^2(G';\chi) = |G'|/|Z| = |G|$.

(ii) This is proved similarly. □

1.2. Cocycle and central extension. Let G be a topological group. A *projective* or *spin* representation π of G is defined as a map assigning to each $g \in G$ a linear map $\pi(g)$ on a vector space $V(\pi)$ which satisfies

$$(9) \qquad \pi(g)\pi(h) = r_{g,h}\pi(gh) \quad (g, h \in G),$$

where $r_{g,h} \in \boldsymbol{C}^{\times} := \{a \in \boldsymbol{C}\,;\, a \neq 0\}$. The function $r_{g,h}$ on $G \times G$ is called the *factor set* associated to π. If we replace $\pi(g)$ by its scalar multiple $\pi'(g) := \lambda_g\pi(g)$, then $\pi'(g)\pi'(h) = r'_{g,h}\pi'(gh)$ with $r'_{g,h} = (\lambda_g\lambda_h/\lambda_{gh})r_{g,h}$. When π is unitary $r_{g,h} \in \boldsymbol{T}^1 := \{a \in \boldsymbol{C}\,;\, |a| = 1\}$.

On the other hand, a function $r_{g,h} \in \boldsymbol{C}^{\times}$ $\big((g,h) \in G \times G\big)$ is called a 2-cocycle of G (with values in $\boldsymbol{C}^{\times}$) if it satisfies

$$(10) \qquad r_{g,h}r_{gh,k} = r_{g,hk}r_{h,k} \quad (g, h, k \in G).$$

Defining that $r_{g,h}$ is equivalent to $r'_{g,h}$ given above, we have the second cohomology group $H^2(G, \boldsymbol{C}^{\times})$ with multiplication product, called as *Schur multiplier* of G.

We assume that a cocycle $r_{g,h}$ is $\boldsymbol{T}^1$-valued, continuous and normalized as $r_{e,e} = 1$. Let Z be the closed subgroup of $\boldsymbol{T}^1$ generated by the set of values $r_{g,h}$. Then, since Z is closed, we have only the following two cases:

Case 1. $Z = \langle e^{2\pi i/n}\rangle \cong \boldsymbol{Z}_n$, cyclic group of order n,

Case 2. $Z = \boldsymbol{T}^1$.
Starting from a cocycle $r_{g,h}$, we can define a central extension G' of G as follows.

Lemma 1.2. *Let G be a topological group, and $r_{g,h}$ $(g, h \in G)$ a cocycle and $Z \subset \boldsymbol{T}^1$ be the closed subgroup generated by the set of values $r_{g,h}$.*

(i) *Introduce in the set $Z \times G$ the following product rule*

$$(z,g)(z',h) := (zz'r_{g,h},\, gh) \quad (z, z' \in Z,\ g, h \in G). \tag{11}$$

Then, we get a central extension G' of G by Z as in (8).

(ii) *Let π be a projective (or spin) representation of G whose factor set is $r_{g,h}$. Then it can be lifted up to a linear representation π' of G' acting on the same representation space $V(\pi)$ in such a way that $\pi'\big((z,g)\big) := z\,\pi(g)$ for $(z,g) \in G'$.*

Proof. (i) Thanks to (10), the associative law holds for the product. Moreover $(z,g)^{-1} = \big(z^{-1}(r_{g,g^{-1}})^{-1}, g^{-1}\big)$.

(ii) By calculation, we have $\pi'\big((z,g)(z',h)\big) = \pi'\big((z,g)\big)\pi'\big((z',h)\big)$. □

We say that the central extension G' in (i) is *associated* to the cocycle $r_{g,h}$, and that the representation π' in (ii) is called a *spin* representation of G' (and also a *spin* representation of G). When we apply this lemma later in §2, it is for a finite group such as $S([\rho])$, and so the central subgroup Z is finite in that case (cf. §2).

1.3. Central character of G' and spin representation of G.

Let G' be a central extension of G by a closed central subgroup Z as in (8). Take a section $s : G \to S_G \subset G'$ for the canonical homomorphism $G' \to G$. Then, for $g, h \in G$, we have $s(g)s(h) = z_{g,h}s(gh)$ with a $z_{g,h} \in Z$.

Lemma 1.3. *For a representation Π of G' of spin type $\chi \in \widehat{Z}$, put*

$$\pi(g) := \Pi\big(s(g)\big) \quad (g \in G). \tag{12}$$

Then, π is a spin representation of G with factor set $t_{g,h} = \chi\big(z_{g,h}\big)$.

Lemma 1.4. *Let G be a compact group and $r_{g,h}$ a continuous cocycle of G with values in $\boldsymbol{T}^1$. Take a central extension G' of G associated to $r_{g,h}$. Then there exist unitary IRs of G' of a certain spin type χ (resp. χ_0) such that $t_{g,h} = \chi(z_{g,h})$ is equal to $r_{g,h}$ (resp. $r_{g,h}^{-1}$).*

Proof. Let Z be a closed subgroup of $\boldsymbol{T}^1$ generated by the set of values $r_{g,h}$. Let χ (resp. χ_0) be the character of Z given by $Z \ni z \to z \in \boldsymbol{T}^1$ (resp. $Z \ni z \to \overline{z} = z^{-1}$). Take a section $s : G \ni g \to (1, g) \in G'$ ($= Z \times G$ as set). Then, take IRs Π of G' of spin type χ (resp. χ_0), whose existence is guaranteed by Lemma 1.1. Putting $\pi(g) = \Pi\big(s(g)\big)$ $(g \in G)$, we have by calculation

$$\begin{aligned}\pi(g)\pi(h) &= \Pi\big(r_{g,h}s(gh)\big) = \chi(r_{g,h})\pi(gh) = r_{g,h}\,\pi(gh)\\ &\qquad \big(\text{resp. } = \chi_0(r_{g,h})\pi(gh) = r_{g,h}^{-1}\,\pi(gh)\big).\end{aligned}$$ □

Let π and π'' be spin representations of G with factor sets $r_{g,h}$ and $r''_{g,h}$ respectively. Then the tensor product $\pi \otimes \pi''$ is a spin representation with factor set $r_{g,h}r''_{g,h}$. Therefore, if $r''_{g,h}$ is the inverse of $r_{g,h}$, that is, $r''_{g,h} = r_{g,h}^{-1}$ $(g, h \in G)$, then $\pi \otimes \pi''$ is non-spin or is reduced from G' to a linear representation of the base group G.

2 Case of semidirect product groups

Let G be a compact group of semidirect product type $G = U \rtimes S$, where U is a compact group, normal in G, and S is a finite group. Here the action of $s \in S$ on $u \in U$ is denoted by $s(u)$.

Take an IR ρ of U and consider its equivalence class $[\rho] \in \widehat{U}$. Every $s \in S$ acts on ρ as ${}^s\rho(u) := \rho(s^{-1}(u))$ $(u \in U)$, and on equivalent classes as $[\rho] \to [{}^s\rho]$. Denote by $\widehat{U}/S$ the set of S-orbits in the dual $\widehat{U}$ of U.

Take a stationary subgroup $S([\rho])$ of $[\rho]$ in S, that is, $S([\rho]) = \{s \in S\,;\, {}^s\rho \cong \rho\}$. Put $H := U \rtimes S([\rho])$. For $s \in S([\rho])$, we determine explicitly an intertwining operator $J_\rho(s)$ as

$$\rho(s(u)) = J_\rho(s)\,\rho(u)\,J_\rho(s)^{-1} \quad (u \in U). \tag{13}$$

Then it is determined up to a non-zero scalar factor. Hence we have a projective representation $S([\rho]) \ni s \mapsto J_\rho(s)$. Let $\alpha_{s,t}$ be its factor set given as

$$J_\rho(s)J_\rho(t) = \alpha_{s,t}\,J_\rho(st) \quad (s,t \in S([\rho])).$$

Let $S([\rho])'$ be a central extension of $S([\rho])$ associated to the cocycle $\alpha_{s,t}$:

$$1 \longrightarrow Z \longrightarrow S([\rho])' \xrightarrow{\Phi_S} S([\rho]) \longrightarrow 1 \quad \text{(exact)},$$

where Φ_S denotes the canonical homomorphism. Then, by Lemma 1.4, J_ρ can be lifted up to a linear representation J'_ρ of $S([\rho])'$. Put $H' := U \rtimes S([\rho])'$ with the action $s'(u) := s(u)$, $s' \in S([\rho])'$, $s = \Phi_S(s')$. Put also

$$\pi^0((u,s')) := \rho(u) \cdot J'_\rho(s') \quad (u \in U,\ s' \in S([\rho])'),$$

then $\pi^0 = \rho \cdot J'_\rho$ is an IR of H'. Take an IR π^1 of $S([\rho])'$ and consider it as a representation of H' through the homomorphism $H' \to S([\rho])' \cong H'/U$, and consider inner tensor product $\pi := \pi^0 \boxdot \pi^1$ as a representation of H'. Let the factor set of π^1, viewed as a spin representation of the base group $S([\rho])$, be $\beta_{s,t}$, then that of π is $\alpha_{s,t}\,\beta_{s,t}$.

To get an IR of G, we pick up π^1 with the factor set $\beta_{s,t} = \alpha_{s,t}^{-1}$ (this is possible by Lemmas 1.1 and 1.4). Then π becomes a linear representation of the base group $H = U \rtimes S([\rho])$. Thus we obtain a representation of G by inducing it up as

$$\Pi(\pi^0,\pi^1) := \mathrm{Ind}_H^G\,\pi = \mathrm{Ind}_H^G(\pi^0 \boxdot \pi^1). \tag{14}$$

Lemma 2.1. *Let ρ be an IR of U, and J'_ρ of $S([\rho])'$ and $\pi^0 = \rho \cdot J'_\rho$ of $H' = U \rtimes S([\rho])'$ be as above. Let π^1 and π^1_o be IRs of $S([\rho])'$, mutually inequivalent, with the factor set inverse to that of J_ρ. Then $\pi = \pi^0 \boxdot \pi^1$ and $\pi_o := \pi^0 \boxdot \pi^1_o$ are irreducible and mutually inequivalent.*

3 Character and irreducibility of $\Pi(\pi^0,\pi^1)$

3.1. Character of $\Pi(\pi^0,\pi^1)$. Put $\Pi = \Pi(\pi^0,\pi^1)$, and let χ_Π be the character of Π. Since $\Pi = \mathrm{Ind}_H^G\pi$, we have the following expression of χ_Π from the general formula for induced representations:

$$\chi_\Pi(g) = \int_{H\backslash G} \chi_\pi(kgk^{-1})\,d\nu_{H\backslash G}(\dot{k}), \tag{15}$$

where the character χ_π of π is extended from H to G by putting 0 outside H, and $\nu_{H\backslash G}$ is the invariant measure on $H\backslash G$ giving mass 1 to each point, and $\dot{k} = Hk$. Since $H\backslash G \cong S([\rho])\backslash S$ is finite, (15) is rewritten, using the the normalized Haar measure μ_G on G, as

$$\chi_\Pi(g) = |H\backslash G| \int_G \chi_\pi(kgk^{-1})\,d\mu_G(k). \tag{16}$$

Note that, for $(u,s) \in H = U \rtimes S([\rho])$, $\chi_\pi((u,s)) = \chi_{\pi^0}((u,s'))\,\chi_{\pi^1}(s')$, with a preimage $s' \in S([\rho])'$ of s : $s = \Phi_S(s')$.

3.2. Irreducibility of $\Pi(\pi^0, \pi^1)$.

Theorem 3.1. *Let $G = U \rtimes S$ with U compact and S finite. Then the induced representation $\Pi(\pi^0,\pi^1) = \mathrm{Ind}_H^G(\pi^0 \boxdot \pi^1)$ of G in (14) is irreducible.*

To prove this, we utilize the following lemma.

Lemma 3.2. *Let ρ_o be an IR of U, and define an IR $\pi_o^0 := \rho_o \cdot J'_{\rho_o}$ of $H'_o := U \rtimes S([\rho_o])'$ similarly as $\pi^0 = \rho \cdot J'_\rho$ of $H' = U \rtimes S([\rho])'$. Assume that ρ_o is not equivalent to ρ. Then, for any $s' \in S([\rho])'$, $s'' \in S([\rho_o])'$,*

$$\int_U \chi_{\pi^0}((u,s'))\,\overline{\chi_{\pi_o^0}((u,s''))}\,d\mu_U(u) \;=\; 0. \tag{17}$$

Proof. Note that the character $\chi_{\pi^0}((u,s')) = \mathrm{tr}(\rho(u)J'_\rho(s'))$ is, as a function in $u \in U$, a linear combination of matrix elements of ρ. Similarly $\chi_{\pi_o^0}((u,s'')) = \mathrm{tr}(\rho_o(u)J'_{\rho_o}(s''))$ is a linear combination of matrix elements of ρ_o. On the other hand, any matrix element of ρ is orthogonal in $L^2(U)$ to any such one of ρ_o. Hence the assertion of the lemma follows. □

Proof of Theorem 3.1. Put $\Pi = \Pi(\pi^0,\pi^1)$. Note that Π is irreducible if and only if

$$\|\chi_\Pi\|^2 = \int_G |\chi_\Pi(g)|^2\,d\mu_G(g) = 1. \tag{18}$$

Then it is enough for us to calculate the integral $\int_G |\chi_\Pi(g)|^2\,d\mu_G(g)$. It is equal to

$$\begin{aligned}
&|H\backslash G|^2 \int_G \iint_{G\times G} \chi_\pi(k_1 g k_1^{-1})\,\overline{\chi_\pi(k_2 g k_2^{-1})}\,d\mu_G(k_1)\,d\mu_G(k_2)\,d\mu_G(g)\\
&= |H\backslash G|^2 \int_G \int_G \chi_\pi(g)\,\overline{\chi_\pi(kgk^{-1})}\,d\mu_G(k)\,d\mu_G(g)\\
&= \int_H \int_{H\backslash G} \chi_\pi(h)\,\overline{\chi_\pi(khk^{-1})}\,d\nu_{H\backslash G}(\dot{k})\,d\mu_H(h) \;=: I_\pi \text{ (put)}.
\end{aligned}$$

Take a complete set of representatives of $H\backslash G \cong S([\rho])\backslash S$ as $\{s_q \in S\,;\, q \in Q\}$ with $s_{q_0} = e$. Then

$$I_\pi = \sum_{q\in Q} \int_H \chi_\pi(h)\,\overline{\chi_\pi(s_q h s_q^{-1})}\,d\mu_H(h). \tag{19}$$

On the other hand, for $h' = (u,s') \in H' = U \rtimes S([\rho])'$ with $h = (u,s), s = \Phi_S(s')$, we have $s_q h s_q^{-1} = (s_q u s_q^{-1}, s_q s s_q^{-1})$. Hence

$$\left\{\begin{aligned}
\chi_\pi(h) &= \mathrm{tr}(\rho(u)J'_\rho(s'))\cdot\chi_{\pi^1}(s'),\\
\chi_\pi(s_q h s_q^{-1}) &= \mathrm{tr}(\rho(s_q u s_q^{-1})J'_\rho(s'_q s' s'^{-1}_q))\cdot\chi_{\pi^1}(s'_q s' s'^{-1}_q),
\end{aligned}\right. \tag{20}$$

where $s'_q \in S([\rho])'$ is a preimage of s_q: $\Phi_S(s'_q) = s_q$. For any $s_q, q \neq q_0$, since $s_q \notin S([\rho])$, IR ${}^{s_q^{-1}}\rho$ is not equivalent to ρ, where $({}^{s_q^{-1}}\rho)(u) = \rho(s_q u s_q^{-1})$. Therefore, taking into account $d\mu_H(h) = d\mu_U(u)\,d\mu_{S([\rho])}(s)$ for $h = (u,s) \in U \rtimes S([\rho])$, and function form in (20), we can apply Lemma 3.2 to the integral term in (19) for $q \neq q_0$, and get $= 0$. Thus, by applying Lemma 2.1 to the case of $q = q_0$, we obtain, as is desired,

$$I_\pi = \int_H \chi_\pi(h)\,\overline{\chi_\pi(h)}\,d\mu_H(h) \;=\; 1. \qquad \square$$

3.3. Orthogonality of characters χ_Π. For $G = U \rtimes S$, let $\{\rho_i\ ;\ \text{IR of } U,\ i \in I_{U;S}\}$ be a complete set of representatives for $\widehat{U}/S$, and for each $i \in I_{U;S}$, let $\{\pi^1_{i,j}\ ;\ j \in J_i\}$ be a complete set of representatives of equivalence classes of IRs of $S([\rho_i])'$ with factor set inverse to that of J_{ρ_i}. Put $H_i := U \rtimes S([\rho_i])$, $H'_i := U \rtimes S([\rho_i])'$, $\pi_{i,j} = \pi^0_i \boxdot \pi^1_{i,j}$, and $\Pi_{i,j} = \Pi(\pi^0_i, \pi^1_{i,j}) = \mathrm{Ind}^G_{H_i} \pi_{i,j}$. Define a set of IRs of G as

$$\Omega(G) := \{\Pi_{i,j} := \Pi(\pi^0_i, \pi^1_{i,j})\ ;\ i \in I_{U,S},\ j \in J_i\}. \tag{21}$$

Theorem 3.3. *For characters $\chi_{\Pi_{i,j}}$, $\Pi_{i,j} \in \Omega(G)$, there hold the following orthogonality relations in $L^2(G)$:*

$$\langle \chi_{\Pi_{i,j}}, \chi_{\Pi_{i',j'}} \rangle_{L^2(G)} = \begin{cases} 1 & \text{if } (i,j) = (i',j'), \\ 0 & \text{if } (i,j) \neq (i',j'), \end{cases} \tag{22}$$

Proof. The case of $(i,j) = (i',j')$ was proved in the proof of Theorem 3.1. Assume $(i,j) \neq (i',j')$, and put $I^{i,j}_{i',j'} := \langle \chi_{\Pi_{i,j}}, \chi_{\Pi_{i',j'}} \rangle_{L^2(G)}$. Then, as in the proof of Theorem 3.1, we have

$$I^{i,j}_{i',j'} = |H_i\backslash G| \cdot |H_{i'}\backslash G| \int_G \int_G \chi_{\pi_{i,j}}(g)\, \overline{\chi_{\pi_{i',j'}}(kgk^{-1})}\, d\mu_G(k)\, d\mu_G(g).$$

Take a complete set of representatives of $H_{i'}\backslash G \cong S([\rho_{i'}])\backslash S$ as $\{s_q \in S\ ;\ q \in Q\}$ with $s_{q_0} = e$. Then

$$I^{i,j}_{i',j'} = \sum_{q \in Q} \int_{H_i} \chi_{\pi_{i,j}}(h)\, \overline{\chi_{\pi_{i',j'}}(s_q h s_q^{-1})}\, d\mu_{H_i}(h). \tag{23}$$

(1) For $i = i'$, similarly as the reasoning after (19), there remains by Lemma 3.2 only the term for $q = q_0$: $I^{i,j}_{i,j'} = \int_{H_i} \chi_{\pi_{i,j}}(h)\, \overline{\chi_{\pi_{i,j'}}(h)}\, d\mu_{H_i}(h)$. By Lemma 2.1, we know that $\pi_{i,j}$ is irreducible, and that $\pi_{i,j} \not\cong \pi_{i,j'}$. Therefore $I^{i,j}_{i,j'} = \delta_{j,j'}$, as is desired.

(2) Assume that $i \neq i'$. Since S-orbits of $[\rho_i]$ and $[\rho_{i'}]$ are different, there exists no s_q such that ${}^{s_q^{-1}}\rho_{i'}$ is equivalent to ρ_i. In the sum over $q \in Q$ in (23), note that $d\mu_{H_i}(h) = d\mu_U(u)\, d\mu_{S[\rho_i]}(s)$ for $h = (u,s)$, $u \in U$, $s \in S([\rho_i])$, and apply Lemma 3.2 again, we see that the integral for any q is equal to 0. So $I^{i,j}_{i,j'} = 0$, as is desired. □

Corollary 3.4. *The set $\Omega(G)$ in (21) of IRs of G consists of mutually inequivalent IRs.*

4 Completeness of the set $\Omega(G)$ of IRs

Let us prove that the set $\Omega(G)$ in (21) of IRs $\Pi_{i,j} = \Pi(\pi^0_i, \pi^1_{i,j})$ is complete, or that our method of induced representations gives essentially all IRs of G.

Theorem 4.1. *Let $G = U \rtimes S$ be such that U is compact and S is finite. Let $\Omega(G)$ be the set of IRs of G defined in (21). Then $\Omega(G)$ gives a complete set of representatives of the dual $\widehat{G}$.*

For the proof, first note $\dim \Pi_{i,j} = \dim \pi_{i,j} \cdot |H_i\backslash G| = \dim \rho_i \cdot \dim \pi^1_{i,j} \cdot |S([\rho_i])\backslash S|$. Recall that $\{\pi^1_{i,j},\ j \in J_i\}$ is a complete set of representatives of spin representations of $S([\rho_i])'$, viewed from the base group $S([\rho_i])$, of a fixed factor set (the inverse of that of J_ρ). Then we have, by Lemma 1.1 (i), $\sum_{j \in J_i} (\dim \pi^1_{i,j})^2 = |S([\rho_i])|$, whence

$$\sum_{j \in J_i} \big(\dim \Pi_{i,j}\big)^2 = (\dim \rho_i)^2 \cdot |S([\rho_i])\backslash S| \times |S|. \tag{24}$$

4. Completeness of the set $\Omega(G)$ of IRs

4.1. Proof in the case where G is finite. Assume that $G = U \rtimes S$ is finite. Under Theorem 3.3, to prove the completeness, it is enough to establish the equality:

$$\sum_{\Pi_{i,j} \in \Omega(G)} \left(\dim \Pi_{i,j}\right)^2 = |G|. \tag{25}$$

Note that $(\dim \rho_i)^2 \cdot |S([\rho_i])\backslash S|$ is equal to the sum of $(\dim \rho)^2$ over $[\rho]$ in the S-orbit of $[\rho_i]$. Since $\{\rho_i\,;\, I_{U;S}\}$ is a complete set of representatives of $\widehat{U}/S$, we have

$$\sum_{i \in I_{U;S}} (\dim \rho_i)^2 \cdot |S([\rho_i])\backslash S| = \sum_{[\rho]\in \widehat{U}} (\dim \rho)^2 = |U|.$$

By (24), this gives the desired equality (25), because $|U| \cdot |S| = |G|$.

4.2. Proof in the case where G is compact. Let $G = U \rtimes S$ be with U compact and S finite. In this case, to prove the completeness of the set $\Omega(G)$ of $\Pi_{i,j}$'s, first we give the following lemma, which corresponds to Lemma 1.1 for G finite.

Lemma 4.2. *Let ρ be an IR of U. Then, the number of equivalence classes $[\Pi] \in \widehat{G}$ of IRs Π of G such that $\Pi|_U$ contains ρ, or $\Pi|_U \supset \rho$, is finite, and*

$$\sum_{[\Pi]\in\widehat{G}:\, \Pi|_U \supset \rho} \left(\dim \Pi\right)^2 = (\dim \rho)^2 \cdot |S([\rho])\backslash S| \cdot |S|. \tag{26}$$

Proof. Denote by $\mathcal{M}_\rho(G)$ the space spanned by matrix elements of $\mathrm{Ind}_U^G \rho$. Then it is a direct sum of spaces $\mathcal{M}(\Pi)$ spanned by matrices of Π, which appear in $\mathrm{Ind}_U^G \rho$ or $\left[\mathrm{Ind}_U^G \rho : \Pi\right] > 0$. By Frobenius reciprocity law $\left[\Pi|_U : \rho\right] = \left[\mathrm{Ind}_U^G \rho : \Pi\right]$, the last condition is equivalent to $\left[\Pi|_U : \rho\right] > 0$, that is, $\Pi|_U \supset \rho$. Hence we obtain

$$\dim \mathcal{M}_\rho(G) = \sum_{[\Pi]\in\widehat{G}:\, \Pi|_U\supset\rho} \left(\dim \Pi\right)^2. \tag{27}$$

On the other hand, the space $V\left(\mathrm{Ind}_U^G \rho\right)$ is spanned by $V(\rho)$-valued functions f on G such that $f(ug) = \rho(u)f(g)$ $(u \in U,\ g \in G)$. Therefore f corresponds 1-1 way to $\varphi := f|_S$ in $\mathcal{F}\left(S; V(\rho)\right)$, the space of $V(\rho)$-valued L^2-functions on S for which the norm is $\|\varphi\|^2 = \int_S \|\varphi(s)\|^2_{V(\rho)} d\mu_S(s)$, where $\|\cdot\|_{V(\rho)}$ denotes the norm in $V(\rho)$. Denote by Π_ρ the realization of $\mathrm{Ind}_U^G \rho$ on $\mathcal{F}\left(S; V(\rho)\right)$. Note that, for $s \in S$ and $g_0 = (u_0, s_0) \in U \rtimes S$, we have $s g_0 = (e, s) g_0 = (s u_0 s^{-1}, s s_0)$, and so

$$\Pi_\rho(g_0)\varphi(s) = \rho(s u_0 s^{-1})\left(\varphi(s s_0)\right). \tag{28}$$

The space $\mathcal{F}\left(S; V(\rho)\right)$ is spanned by functions of the form $\varphi_{v,\psi}(s) := v \cdot \psi(s)$ $(s \in S)$, where $v \in V(\rho)$, $\psi \in L^2(S)$. Take $\varphi_1, \varphi_2 \in \mathcal{F}\left(S; V(\rho)\right)$ as $\varphi_i(s) = v_i \cdot \psi_i(s)$ $(s \in S)$ with $v_i \in V(\rho)$ and $\psi_i \in L^2(S)$. Calculate the matrix element for Π_ρ as

$$\begin{aligned}\left\langle \Pi_\rho(g_0)\varphi_1, \varphi_2\right\rangle &= \int_S \left\langle \Pi_\rho(g_0)\varphi_1(s), \varphi_2(s)\right\rangle_{V(\rho)} d\mu_S(s) \\ &= \int_S \left\langle \rho(s u_0 s^{-1}) v_1, v_2\right\rangle_{V(\rho)} \psi_1(s s_0)\, \overline{\psi_2(s)}\, \mu_S(s) \quad \left(=: F(g_0) \text{ (put)}\right).\end{aligned}$$

For $t \in S$, denote by δ_t the delta functions on S given as $\delta_t(s) = 1$ or 0 according as $s = t$ or not. Put $\psi_i = \delta_{t_i}$ for $t_i \in S$. Then

$$F(g_0) = |S|^{-1} \cdot \left\langle \rho(t_2 u_0 t_2^{-1}) v_1, v_2\right\rangle_{V(\rho)} \cdot \delta_{t_2^{-1}t_1}(s_0). \tag{29}$$

Here, the first factor, as a function in $u_0 \in U$, spans the space $\mathcal{M}([{}^{t_2^{-1}}\rho])$ of matrix elements of an IR ${}^{t_2^{-1}}\rho$ of U. The second factor, as a function in $s_0 \in S$, $\delta_t(s_0)$ with $t = t_2^{-1}t_1$ spans the space $\mathcal{F}(S)$ of all functions on S. Therefore we obtain

$$\dim \mathcal{M}_\rho(G) = \sum_{\text{different } [{}^s\rho]} \dim \mathcal{M}([{}^s\rho]) \cdot \dim \mathcal{F}(S) \tag{30}$$

$$= \sum_{[^s\rho]} (\dim {}^s\rho)^2 \cdot |S| = (\dim \rho)^2 \cdot |S([\rho])\backslash S| \cdot |S|.$$

From (27) and (30), we obtain the desired equality (26). □

Applying Lemma 4.2 above, we see that the completeness of the set $\Omega(G)$ of $\Pi_{i,j}$ is equivalent to the following equality: for each $i \in I_{U;S}$,

$$\sum_{j \in J_i} \big(\dim \Pi_{i,j}\big)^2 = (\dim \rho_i)^2 \cdot |S([\rho_i])\backslash S| \cdot |S|. \tag{31}$$

However this is already proved in (24).

Remark 4.1. In his Chicago lecture note [Macl], G.W. Mackey discussed construction of irreducible representation or factor representations of semidirect product groups $U \rtimes S$. However the explicit statement such as Theorem in Introduction or Theorem 4.1 in the present paper cannot be found for the case where U is compact and S is finite. It seems that his discussion is beyond this classical case.

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5 Appendix 1. Examples

(Appendix 1, Appendix 2 are added for this report)

Example 5.1. Let $\mathfrak{S}_n$, $n \geq 2$, be the n-th symmetric group. Then, as an abstract group, $\mathfrak{S}_n$ is presented by giving a set of generators and a set of fundamental relations as follows. As the set of generators, $\{s_i\,;\, i \in \boldsymbol{I}_{n-1}\}$ with $\boldsymbol{I}_k := \{1, 2, \ldots, k\}$, and as the set of fundamental relations:

$$(32) \qquad \begin{cases} s_i^{\,2} = e & (i \in \boldsymbol{I}_{n-1}), \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & (i \in \boldsymbol{I}_{n-2}), \\ s_i s_j = s_j s_i & (i, j \in \boldsymbol{I}_{n-1},\ |i-j| \geq 2), \end{cases}$$

where e denotes the identity element of the group. The Schur multiplier $H^2(\mathfrak{S}_n, \boldsymbol{C}^\times)$ is trivial for $n = 2, 3$, and $\boldsymbol{Z}_2$ for $n \geq 4$. A central extension $\widetilde{\mathfrak{S}}_n$ by a central subgroup $Z_1 := \{e, z_1\}$, $z_1^{\,2} = e$, is given as follows: as a set of generators: $\{z_1,\ r_i\ (i \in \boldsymbol{I}_{n-1})\}$, and as a set of fundamental relations:

$$(33) \qquad \begin{cases} z_1^{\,2} = e, \quad r_i z_1 = z_1 r_i & (i \in \boldsymbol{I}_{n-1}), \\ r_i^{\,2} = e & (i \in \boldsymbol{I}_{n-1}), \\ r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1} & (i \in \boldsymbol{I}_{n-2}), \\ r_i r_j = z_1 r_j r_i & (i, j \in \boldsymbol{I}_{n-1},\ |i-j| \geq 2), \end{cases}$$

and the canonical homomorphism $\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ is given by $\Phi_{\mathfrak{S}}(z_1) = e$, $\Phi_{\mathfrak{S}}(r_i) = s_i$ $(i \in \boldsymbol{I}_{n-1})$. For $n \geq 4$, the double covering group $\widetilde{\mathfrak{S}}_n$ is one of representation groups of $\mathfrak{S}_n$, given by Schur and denoted by $\mathfrak{T}'_n$ [Sch2, §3].

Let $\mathfrak{A}_n$ be the n-th alternating group, and $\widetilde{\mathfrak{A}}_n := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_n)$ the full inverse image of $\mathfrak{A}_n \subset \mathfrak{S}_n$. Then the representation group $\mathfrak{B}_n$ of $\mathfrak{A}_n$ is unique, and is given by $\widetilde{\mathfrak{A}}_n$ for $n \geq 4, n \neq 6, 7$, and is a 6-times covering group of $\mathfrak{A}_n$ for $n = 6, 7$ for which $\widetilde{\mathfrak{A}}_n$ is a quotient group (cf. [Sch2, §5]).

Here we note that, for a finite group G, a central extension G' of G is called a *representation group* of G if (1) any spin representation of G can be obtained from a linear representation of G' as in Lemma 1.3, and (2) among such central extensions of G, the order of G' is minimum. By Schur [Sch1], any finite group G has a finite number of non-isomorphic representation groups, and for every such G', the central subgroup Z in (8) is isomorphic to Schur multiplier $H^2(G, \boldsymbol{C}^\times)$.

Example 5.2. Let $\boldsymbol{Z}_m$ be a cyclic group of order m, for which the product is written multiplicatively. Let y be a fixed generator of $\boldsymbol{Z}_m$, and let $D_n(\boldsymbol{Z}_m)$ be the direct product of n-copies of $\boldsymbol{Z}_m$, so that it is given by presenting a set of generators and a set of fundamental relations as follows. As the set of generators $\{y_j\,;\, j \in \boldsymbol{I}_n\}$ (y_j is the j-th copy of y), and as the set of fundamental relations

$$(34) \qquad \begin{cases} y_j^{\,m} = e & (j \in \boldsymbol{I}_n), \\ y_j y_k = y_k y_j & (j, k \in \boldsymbol{I}_n). \end{cases}$$

The Schur multiplier is given as $H^2(D_n(\boldsymbol{Z}_m), \boldsymbol{C}^\times) \cong \boldsymbol{Z}_m^{\,n(n-1)/2}$, and so $D_n(\boldsymbol{Z}_m)$ has many kinds of non-trivial central extensions.

Assume m is even. Then a central extension $\widetilde{D}_n(\boldsymbol{Z}_m)$ by a central subgroup $Z_2 := \{e, z_2\}$, $z_2^2 = e$, is given as follows: as a set of generators $\{\eta_j\,;\, j \in \boldsymbol{I}_n\}$, and as a set of fundamental relations

$$(35) \qquad \begin{cases} z_2^2 = e, \quad \eta_j z_2 = z_2 \eta_j & (j \in \boldsymbol{I}_n), \\ \eta_j^{\,m} = e & (j \in \boldsymbol{I}_n), \\ \eta_j \eta_k = z_2 \eta_k \eta_j & (j, k \in \boldsymbol{I}_{n-1},\ j \neq k), \end{cases}$$

and the canonical homomorphism $\Phi_D : \widetilde{D}_n(\boldsymbol{Z}_m) \to D_n(\boldsymbol{Z}_m)$ is given by $\Phi_D(z_2) = e$, $\Phi_D(\eta_j) = y_j$ $(j \in \boldsymbol{I}_n)$.

Example 5.3. Consider the dual of the covering group $\widetilde{D}_n(\boldsymbol{Z}_m)$ for m even. Its IRs ρ have two kinds of spin types $\beta_2 = \pm 1$ given as $\rho(z_2) = \beta_2 I$.

The non-spin case $\beta_2 = 1$ is for one-dimensional characters of $D_n(\boldsymbol{Z}_m)$. Put

$$(36)\qquad \begin{cases} \Gamma_n := \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)\ ;\ 0 \le \gamma_j < m\ (j \in \boldsymbol{I}_n)\}, \\ \Gamma_n^0 := \{\gamma = (\gamma_j)_{j\in \boldsymbol{I}_n} \in \Gamma_n\ ;\ 0 \le \gamma_j < m/2\ (j \in \boldsymbol{I}_n)\}. \end{cases}$$

For $\gamma \in \Gamma_n$, define one-dimensional character ζ_γ of $\widetilde{D}_n(\boldsymbol{Z}_m)$ by $\zeta_\gamma(\eta_j) := \omega^{\gamma_j}$, where $\omega := \exp(2\pi i/m)$ is a primitive m-th root of unity.

For the spin case $\beta_2 = -1$, introduce 4 matrices in the unitary group $U(2)$ as

$$(37)\qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and put $Y_j \in U(2^k)$, $j \in \boldsymbol{I}_{2k+1}$, with $k = [n/2]$, as

$$\begin{cases} Y_{2i-1} = c^{\otimes(i-1)} \otimes a \otimes \varepsilon^{\otimes(k-i)} \\ Y_{2i} \ \ \ \ = c^{\otimes(i-1)} \otimes b \otimes \varepsilon^{\otimes(k-i)} \end{cases} \quad (\text{for } i \in \boldsymbol{I}_k),$$
$$Y_{2k+1} = c^{\otimes(k-1)} \otimes c,$$

where $x^{\otimes p}$ denote p-times tensor product of x ($x^{\otimes p}$ is absent if $p = 0$).

Fact 5.1. *Put $E := E_{2^k}$ the identity matrix of degree 2^k, then there hold*

$$(38)\qquad \begin{cases} Y_j^{\,2} &= E & (j \in \boldsymbol{I}_{2k+1}), \\ Y_j Y_l &= -Y_l Y_j & (j, l \in \boldsymbol{I}_{2k+1},\ j \ne l), \\ Y_1 Y_2 \cdots Y_{2k+1} &= i^k E & (i = \sqrt{-1}). \end{cases}$$

For generators $\{z_2,\ \eta_j\ (j \in \boldsymbol{I}_n)\}$ of $\widetilde{D}_n(\boldsymbol{Z}_m)$, put for $\gamma \in \Gamma_n$

$$(39)\qquad P_\gamma(z_2) := -E, \quad P_\gamma(\eta_j) := \zeta_\gamma(\eta_j) Y_j \ \ (j \in \boldsymbol{I}_n).$$

Then P_γ maps the fundamental relations in (35) isomorphically as seen from Fact 5.1 above. Hence P_γ gives a spin representation of $\widetilde{D}_n(\boldsymbol{Z}_m)$. Actually the family of P_γ's covers all equivalence classes of spin IRs of it. To discuss such things, we utilize the character χ_{P_γ} of P_γ given in [HHoH2, Theorem 6.3]. Express $d' \in \widetilde{D}_n(\boldsymbol{Z}_m)$ uniquely in the following form:

$$(40)\qquad d' = z_2^{\,b} \eta_1^{\,a_1} \eta_2^{\,a_2} \cdots \eta_n^{\,a_n}, \quad b = 0, 1,\ \ 0 \le a_j < m\ (j \in \boldsymbol{I}_n).$$

Theorem 5.1. *The character χ_{P_γ} of IR P_γ of $\widetilde{D}_n(\boldsymbol{Z}_m)$ is given as follows.*
(i) *Assume $n \ge 2$ is even. For $d' \in \widetilde{D}_n(\boldsymbol{Z}_m)$ in (40),*

$$\chi_{P_\gamma}(d') = \begin{cases} 2^{[n/2]} \cdot \zeta_\gamma(d')(-1)^b, & \text{if } a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv 0 \pmod 2, \\ 0, & \text{otherwise}. \end{cases}$$

(ii) *Assume $n \ge 3$ odd. For $d' \in \widetilde{D}_n(\boldsymbol{Z}_m)$ in (40),*

$$\chi_{P_\gamma}(d') = \begin{cases} 2^{[n/2]} \cdot \zeta_\gamma(d')(-1)^b, & \text{if } a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv 0 \pmod 2, \\ (2i)^{[n/2]} \cdot \zeta_\gamma(d')(-1)^b, & \text{if } a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv 1 \pmod 2, \\ 0, & \text{otherwise}. \end{cases}$$

Let $j \in \boldsymbol{I}_n$. For a parameter $\gamma = (\gamma_j)_{j \in \boldsymbol{I}_n} \in \Gamma_n$, put $\tau_p \gamma = (\gamma'_j)_{j\in\boldsymbol{I}_n}$ with $\gamma'_p = \gamma_p + m/2 \pmod m$, $\gamma'_j = \gamma_j\ (j \ne p)$.

Corollary 5.2. (i) *Assume $n \ge 2$ is even. Then $P_\gamma \cong P_{\tau_p\gamma}$ ($\gamma \in \Gamma_n,\ p \in \boldsymbol{I}_n$).*

(ii) *Assume $n \geq 3$ odd.* $P_\gamma \cong P_{\tau_p\tau_q\gamma}$ ($\gamma \in \Gamma_n,\ p,q \in \boldsymbol{I}_n, p \neq q$).

When n is even, for $\gamma = \mathbf{0} = (0,0,\ldots,0)$, put $P := P_{\mathbf{0}}$, and when n is odd, put $P_+ := P_{\mathbf{0}}$, $P_- := P_{\tau_n\mathbf{0}}$ with $\tau_n\mathbf{0} = (0,\ldots,0,m/2)$. Then, as a complete set of representatives of equivalence classes of spin IRs, we can take the following:

$$
(41)\qquad \begin{array}{ll} \{P_\gamma = \zeta_\gamma \otimes P\ ;\ \gamma \in \Gamma_n^0\}, & \text{for } n \geq 2 \text{ even}, \\ \{P_\gamma = \zeta_\gamma \otimes P_+,\ P_{\tau_n\gamma} = \zeta_\gamma \otimes P_-\ ;\ \gamma \in \Gamma_n^0\}, & \text{for } n \geq 3 \text{ odd}. \end{array}
$$

Example 5.4. Consider a semidirect product group $G = U \rtimes S$ with $U = \widetilde{D}_n(\boldsymbol{Z}_m)$ and $S = \mathfrak{S}_n$,where $\mathfrak{S}_n$-action is given as $\sigma\eta_j = \eta_{\sigma(j)}$ ($\sigma \in \mathfrak{S}_n,\ j \in \boldsymbol{I}_n$). We see from (35) that this actually gives an action. Take an IR $\rho = P_\gamma$ of $\widetilde{D}_n(\boldsymbol{Z}_m)$. We determine the stationary subgroup $\mathfrak{S}_n([\rho])$. Then we calculate explicitly intertwining operators $J_\rho(s)$ for $s \in \mathfrak{S}_n([\rho])$.

Note that, for $\sigma \in \mathfrak{S}_n$, $({}^{\sigma^{-1}}P_\gamma)(d') = P_\gamma(\sigma(d'))$, and that ${}^{\sigma^{-1}}P_\gamma \cong P_\gamma$ if and only if their characters conincide. Then we can determine, from the character formula in Theorem 5.1, the stationary subgroup $\mathfrak{S}_n([\rho])$ of $[\rho]$ in $\mathfrak{S}_n$ for $\rho = P_\gamma,\ P_{\tau_n\gamma}$, as follows.

Proposition 5.3. *Let $\gamma \in \Gamma_n^0$.*
(i) *When $n \geq 2$ is even,* $\mathfrak{S}_n([P_\gamma]) = \{\sigma \in \mathfrak{S}_n\ ;\ \sigma\gamma = \gamma\}$.
(ii) *When $n \geq 3$ is odd,* $\mathfrak{S}_n([P_\gamma]) = \mathfrak{S}_n([P_{\tau_n\gamma}]) = \{\sigma \in \mathfrak{A}_n\ ;\ \sigma\gamma = \gamma\}$.

Denote the subgroups in (i) and (ii) above by $\mathfrak{S}_n(\gamma)$ and $\mathfrak{A}_n(\gamma)$ respectively.

Note that $\sigma P_\gamma \cong P_{\sigma\gamma}$ ($\sigma \in \mathfrak{S}_n$) in case n is even, and that $\sigma P_\gamma \cong P_{\sigma\gamma}$ ($\sigma \in \mathfrak{A}_n$) and $\sigma P_\gamma \cong P_{\tau_n\sigma\gamma}$ ($\sigma \notin \mathfrak{A}_n$) in case n is odd. Then we see that, among $\sigma\gamma$ there exists a standard element (denoted again by γ) such that

$$
(42)\qquad \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n.
$$

Hence we have a set of representatives $\{P_\gamma\}$ in case n is even (resp. $\{P_\gamma,\ P_{\tau_n\gamma}\}$ in case n is odd) of $\mathfrak{S}_n$-orbits in spin dual of $\widetilde{D}_n(\boldsymbol{Z}_m)$ for which $\gamma \in \Gamma_n^0$ satisfies the condition (42). An advantage of this condition is that $\mathfrak{S}_n([\rho])$ for such a representative $[\rho]$ is generated by simple reflections if n is even (resp. products of two simple reflections if n is odd) and so the intertwining operators $J_\rho(s)$ can be given using simple reflections if n is even (resp. products of two such ones if n is odd).

Lemma 5.4. *Put* $\nabla_n(z_2) := -E$, $\nabla_n(r_i) := \frac{1}{\sqrt{2}}(Y_i - Y_{i+1})$ *for* $i \in \boldsymbol{I}_{n-1}$. *Then*

$$
(43)\qquad \begin{cases} \nabla_n(r_i)^2 = E \quad (i \in \boldsymbol{I}_{n-1}), \\ \nabla_n(r_i)\nabla_n(r_{i+1}) + \nabla_n(r_{i+1})\nabla_n(r_i) + E = O \quad (i \in \boldsymbol{I}_{n-2}), \\ \nabla_n(r_j)\nabla_n(r_k) = -\nabla_n(r_k)\nabla_(r_j) \quad (j,k \in \boldsymbol{I}_{n-1},\ j \neq k). \end{cases}
$$

Moreover, for $j \in \boldsymbol{I}_n$, $\nabla_n(r_i)Y_j\nabla_n(r_i)^{-1} = Y_{s_i(j)}$.

From the first assertion, we can prove that the correspondence $z_2 \to \nabla_n(z_2)$, $r_i \to \nabla_n(r_i)$ ($i \in \boldsymbol{I}_{n-1}$) represents the set of fundamental relations (33) for the covering group $\widetilde{\mathfrak{S}}_n$, and so it defines a spin representation of $\mathfrak{S}_n$ (and of $\widetilde{\mathfrak{S}}_n$ in our terminology). Moreover it follows from the second assertion that, if $s_i \in \mathfrak{S}_n(\gamma)$,

$$
({}^{s_i}P_\gamma)(\eta_j) = P_\gamma(s_i(\eta_j)) = P_\gamma(\eta_{s_i(j)}) = \omega^{\gamma_{s_i(j)}}Y_{s_i(j)} = \nabla_n(r_i)P_\gamma(\eta_j)\nabla_n(r_i)^{-1},
$$

and so $J_\rho(s_i) = \lambda_i\nabla_n(r_i)$ with a $\lambda_i \in \boldsymbol{C}^\times$. Thus J_ρ is a spin representation of $\mathfrak{S}_n([\rho])$ for $\rho = P_\gamma$, given by restricting ∇_n on it, and we see that $\pi^0 = \rho \cdot J'_\rho$ is a spin representation.

Thus to obtain (non-spin) IRs of $G = \widetilde{D}_n(\boldsymbol{Z}_m) \rtimes \mathfrak{S}_n$, we should take spin IR π^1 of $\mathfrak{S}_n(\gamma)$ or of $\mathfrak{A}_n(\gamma)$. Here we need a study on twisted central product of double coverings of finite groups and twisted central products of thier spin IRs (cf. [HHo]).

6 Appendix 2. Examples (suite)

Example 6.1. Let $\mathcal{M}(N, C)$ be the algebra of all complex $N \times N$ matrices. When $N = 2^k$, it has a special structure as follows.

Fact 6.1. *The set of matrices of the form* $\{Y_1^{a_1} Y_2^{a_2} \cdots Y_{2k}^{a_{2k}}\,;\ a_j = 0, 1\ (j \in \boldsymbol{I}_{2k})\}$ *gives a linear basis of* $\mathcal{M}(2^k, C)$.

Take $\{Y_j\ (j \in \boldsymbol{I}_{2k})\}$ as a set of generators of the algebra $\mathcal{M}(2^k, C)$, then the corresponding set of fundamental relations are given by

$$Y_j^{\,2} = E \ \ (j \in \boldsymbol{I}_{2k}), \quad Y_j Y_l = -Y_l Y_j \ \ (j \neq l\,;\ j, l \in \boldsymbol{I}_{2k}). \tag{44}$$

Three kinds of actions of permutation groups on $\mathcal{M}(2^k, C)$ are given as follows: put $Y_j' := (-1)^{j-1} Y_j\ (j \in \boldsymbol{I}_n)$, and

$$\text{(6.1.1)} \qquad \text{For } \sigma \in \mathfrak{S}_{2k+1}: \quad \sigma^{(1)}(Y_j') := \mathrm{sgn}(\sigma)\, Y'_{\sigma(j)} \qquad (j \in \boldsymbol{I}_{2k+1});$$

$$\text{(6.1.2)} \qquad \text{For } \sigma \in \mathfrak{S}_{2k+1}: \quad \sigma^{(2)}(Y_j) := \mathrm{sgn}(\sigma)\, Y_{\sigma(j)} \qquad (j \in \boldsymbol{I}_{2k+1});$$

$$\text{(6.1.3)} \qquad \text{For } \sigma \in \mathfrak{S}_{2k}: \quad \sigma^{(3)}(Y_j) := Y_{\sigma(j)} \qquad (j \in \boldsymbol{I}_{2k})$$
$$\text{(accordingly } Y_{2k+1} \to \mathrm{sgn}(\sigma) Y_{2k+1}\text{)},$$

where the upper suffices (1), (2) and (3) are added to distinguish different actions. To see that these formulas actually give actions on the algebra $\mathcal{M}(2^k, C)$, we refer to fundamental relations in (38) for (6.1.1) and (6.1.2), and those in (44) for (6.1.3).

Consequently each of $\mathfrak{S}_{2k+1}$ and $\mathfrak{S}_{2k}$ acts, in its way, on $GL(2^k, C) = \mathcal{M}(2^k, C)^\times$, the group of all invertible elements. On the other hand, since every automorphism of $\mathcal{M}(N, C)$ is inner, we have a regular matrix $J^{(\alpha)}(\sigma)$ ($\alpha = 1 \sim 3$) such that $\sigma^{(\alpha)}(X) = J^{(\alpha)}(\sigma) X J^{(\alpha)}(\sigma)^{-1}$ for $X \in \mathcal{M}(2^k, C)$. Hence, $\det\big(\sigma^{(\alpha)}(X)\big) = \det X$, and so these groups act also on $SL(2^k, C)$ respectively. For each α, the matrix $J^{(\alpha)}(\sigma)$ is determined up to a scalar multiple, and $\sigma \mapsto J^{(\alpha)}(\sigma)$ may give a spin representation of the corresponding group. Actually it is the case and here is the origin of Schur's 'Hauptdarstellung' of $\mathfrak{S}_n$ (cf. [Sch2]). The matrices $J^{(\alpha)}(s_i)$ for simple transpositions s_i ($i \in \boldsymbol{I}_{n-1}$), corresponding to (6.1.1) $\sim$ (6.1.3) respectively, are given as $J^{(\alpha)}(s_i) = c_i^{(\alpha)}\, \nabla^{(\alpha)}(r_i)$ with constants $c_i^{(\alpha)}$, for $\alpha = 1 \sim 3$, where

$$\text{(6.2.1)} \qquad \nabla^{(1)}(r_i) \ := \ \frac{1}{\sqrt{2}}(Y_i' - Y_{i+1}') \qquad (i \in \boldsymbol{I}_{2k})\,;$$

$$\text{(6.2.2)} \qquad \nabla^{(2)}(r_i) \ := \ \frac{1}{\sqrt{2}}(Y_i - Y_{i+1}) \qquad (i \in \boldsymbol{I}_{2k})\,;$$

$$\text{(6.2.3)} \qquad \nabla^{(3)}(r_i) \ := \ \nabla^{(2)}(r_i) \cdot \sqrt{-1}\, Y_{2k+1} \qquad (i \in \boldsymbol{I}_{2k-1})\,.$$

We can prove by calculation that

$$\begin{cases} \nabla^{(1)}(r_i)^2 = E & (i \in \boldsymbol{I}_{2k-1}), \\ \nabla^{(1)}(r_i)\nabla^{(1)}(r_{i+1}) + \nabla^{(1)}(r_{i+1})\nabla^{(1)}(r_i) + E = O & (i \in \boldsymbol{I}_{2k-2}), \\ \nabla^{(1)}(r_i)\nabla^{(1)}(r_j) = -\nabla^{(1)}(r_j)\nabla^{(1)}(r_i) & (i \not\sim j). \end{cases} \tag{45}$$

This implies that the fundamental relations in (33) are isomorphically mapped by $r_i \mapsto \nabla^{(1)}(r_i)$ with $\nabla^{(1)}(z_1) = -E$, and accordingly we see that $\nabla^{(1)}$ gives a spin (unitary) representation of $\widetilde{\mathfrak{S}}_n$, $n = 2k+1$, and that $s \mapsto J^{(1)}(s)$ ($s \in \mathfrak{S}_n$) is a spin (or double-valued) representation of $\mathfrak{S}_n$.

Similar arguments can be given for $\nabla^{(\alpha)}(r_i)$ and $J^{(\alpha)}(s)$ for $\alpha = 2, 3$, and see that $J^{(2)}(s)$ (resp. $J^{(3)}(s)$) is a spin representation of $\mathfrak{S}_n$, $n = 2k+1$ (resp. $n = 2k$).

Example 6.2. The group $SO(n)$ acts on the n-dimensional Euclidean space V_n. Let $\{f_1, f_2, \ldots, f_n\}$ be an orthonormal basis of V_n, then $\mathfrak{S}_n$ acts naturally on V_n by permuting

$f_1, f_2, \ldots, f_n$, and $\mathfrak{S}_n$ is thus contained in $O(n)$ and acts on $SO(n)$ by conjugation. On the other hand, the universal covering group $Spin(n)$ of $SO(n)$ is realized as follows [Che, Chap. 2]. Let $\mathscr{E}_n := \{e_j\,;\ j \in I_n\}$ be the set of generators (over $\boldsymbol{R}$) of Clifford numbers $\mathscr{C}_n$, with the identity element e_0, which is defined by the set of fundamental relations

$$e_j^2 = -e_0 \ (j \in I_n), \qquad e_j e_k = -e_k e_j \ \ (j,k \in I_n,\, j \neq k).$$

Then $\mathfrak{S}_n$ acts on $\mathscr{E}_n$ as $\sigma(e_j) := e_{\sigma(j)}$ (or as $\sigma(e_j) := \mathrm{sgn}(\sigma)e_{\sigma(j)}$) for $j \in I_n$, and so does on $\mathscr{C}_n$ and also on $\mathscr{C}_n^\times$. The covering group $Spin(n)$ is defined as the subgroup of $\mathscr{C}_n^\times$ generated by $\exp(\theta\, e_j e_l) = \cos\theta\, e_0 + \sin\theta\, e_j e_l$ $(j \neq l)$, and it acts on the space $V_n' = \langle e_j\ (j \in I_n)\rangle_{\boldsymbol{R}}$, spanned over $\boldsymbol{R}$ by e_j's, as $V_n' \ni v' \mapsto u'v'u'^{-1} = u(v') \in V_n'$ $(u' \in Spin(n))$, which gives the canonical homomorphism $\Phi_{SO} : Spin(n) \ni u' \mapsto u \in SO(n) \cong Spin(n)/\{\pm e_0\}$, under the natural identification of V_n' and V_n. Hence $\mathfrak{S}_n$ also acts on $Spin(n)$.

The map $e_0 \to E$, $e_j \to \sqrt{-1}\,Y_j$ $(j \in I_n)$ gives a representation of $\mathscr{C}_n$ of dimension 2^k, $k = [n/2]$, and so $\exp(\theta\, e_j e_l) \to \exp\left(-\theta\, Y_j Y_l\right)$ gives a linear representation ρ of the group $Spin(n)$ such that $\rho(-e_0) = -E$, and ρ is a spin representation of $SO(n)$.

Here, one-parameter subgroups $\exp(\theta_p e_{2p-1}e_{2p})$ $(p \in I_k)$, which generate a Cartan subgroup H of $Spin(n)$, are respectively mapped by ρ to

$$v_p(\theta_p) := \exp\left(- \theta_p Y_{2p-1}Y_{2p}\right) = \varepsilon^{\otimes(p-1)} \otimes u_2(\theta_p) \otimes \varepsilon^{\otimes(k-p)}$$

with $\quad u_2(\theta_p) := \mathrm{diag}(e^{-i\theta_p}, e^{i\theta_p}).$

Therefore all weights of ρ are multiplicity-free and of the form $\mu = (\pm 1/2, \pm 1/2, \ldots, \pm 1/2)$. The trace of $\prod_{p\in I_k} v_p(\theta_p)$ gives the character χ_ρ of ρ: for $h = h(\theta_1, \theta_2, \ldots, \theta_k) := \prod_{p\in I_k} \exp(\theta_p e_{2p-1}e_{2p}) \in H$,

$$\chi_\rho(h) = \prod\nolimits_{p\in I_k} \left(e^{-i\theta_p} + e^{i\theta_p}\right). \tag{46}$$

Proposition 6.1. (i) *Suppose $n = 2k+1 \geq 3$ is odd. The spin representation ρ of $SO(n)$ is irreducible, with highest weight $\lambda = (1/2, 1/2, \ldots, 1/2)$ and dimension $2^{[n/2]} = 2^k$. The stationary subgroup of its equivalence class is $\mathfrak{S}_n([\rho]) = \mathfrak{S}_n$.*

(ii) *Suppose $n = 2k \geq 4$ is even. The spin representation ρ of $SO(n)$ is split into two non-equivalent IRs ρ_+ and ρ_- with highest weight $\lambda_\epsilon = (1/2, \ldots, 1/2, \epsilon\, 1/2)$, $\epsilon = \pm$, and dimension $2^{[n/2]-1} = 2^{k-1}$, and with characters*

$$\chi_{\rho_\epsilon}(h) = \sum_{\epsilon_1\epsilon_2\cdots\epsilon_k = \epsilon} e^{\epsilon_1 i\theta_1} e^{\epsilon_2 i\theta_2} \cdots e^{\epsilon_k i\theta_k}. \tag{47}$$

The stationary subgroup in $\mathfrak{S}_n$ is given as $\mathfrak{S}_n([\rho_\pm]) = \mathfrak{A}_n$, and ${}^\sigma(\rho_\pm) = \rho_\mp$ if $\mathrm{sgn}(\sigma) = -1$.

Proof. The image of ρ contains $Y_jY_l = \exp\left(\pi/2\, Y_jY_l\right)$ for all $j \neq l$ $(j, l \in I_n)$.

(i) We see from Fact 1 that $Y_1Y_2\cdots Y_{2k} = i^k\, Y_{2k+1}$, and then from Fact 2, that the set Y_jY_l $(j \neq l)$ generates the total algebra $\mathcal{M}(2^k, \boldsymbol{C})$. This means that ρ is irreducible. The rest of the assertions is proved easily.

(ii) Let $V = \boldsymbol{C}^{2^k}$ be the vector space on which Y_j's act, and v_μ a non-zero weight vector for $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, $\mu_j = \pm 1/2$. Then they give a basis of V. Let V_ϵ $(\epsilon = \pm)$ be the subspace of V spanned by v_μ such that $\mu_1\mu_2\cdots\mu_k = \epsilon 1$, then $V = V_+ \oplus V_-$, and $Y_jV_+ = V_-$. We see from Facts 1 and 2 that the set of Y_jY_l $(j \neq l)$ generates a subalgebra $\mathcal{M}_+$ of $\mathcal{M}(2^k, \boldsymbol{C})$, and so the algebra $\langle \rho(Spin(n))\rangle$ generated by the image of ρ is $\mathcal{M}_+$. Moreover $\mathcal{M}(2^k, \boldsymbol{C})$ is a direct sum of $\mathcal{M}_+$ and $Y_j\mathcal{M}_+$ as vector space. By calculation we see that each V_ϵ is invariant under $\mathcal{M}_+$. In fact, the set $\{Y_jY_{j+1}\,;\ j \in I_{n-1}\}$ generates $\mathcal{M}_+$, and $Y_{2p-1}Y_{2p}$ commutes with $h(\theta_1, \ldots, \theta_k)$, and

$$\left(Y_{2p}Y_{2p+1}\right) h(\theta_1, \ldots, \theta_k) \left(Y_{2p}Y_{2p+1}\right)^{-1}$$

gives parameter change $(\theta_p, \theta_{p+1}) \to (-\theta_{p+1}, -\theta_p)$. The restriction of ρ on V_ϵ is denoted by ρ_ϵ, and its character is given by the formula in (47) since weights of V_ϵ is multiplicity-free. Once we identify the highest weight of ρ_ϵ as $\lambda_\epsilon = (1/2, \ldots, 1/2, \epsilon\, 1/2)$, the rest of assertions can be proved from the property of these IRs. □

Note 6.1. For $h(\theta_1, \theta_2, \ldots, \theta_k) \in H$, the action of Weyl group W_n of $Spin(n)$ is given as follows. If n is odd, then $Spin(n)$ is of type B_n, and W_n is generated by (1) any permutation of $(\theta_p)_{p \in I_k}$, and (2) sign changes of any number of θ_p's. If n is even, then $Spin(n)$ is of type D_n, and W_n is generated by (1) and (2′) sign changes of even number of θ_p's.

When n is odd, the intertwining operator $J_\rho(s)$ $\big(s \in \mathfrak{S}_n([\rho])\big)$ is defined as

$$\rho\big(s(u')\big) = J_\rho(s)\, \rho(u')\, J_\rho(s)^{-1} \quad \big(u' \in Spin(n)\big), \tag{48}$$

and when n is even, similarly for ρ_+ and ρ_-. From (6.1.2)–(6.2.3), we obtain the following.

Theorem 6.2. (i) *Suppose $n = 2k+1 \geq 3$ is odd. The intertwining operators $J_\rho(s)$ $\big(s \in \mathfrak{S}_n([\rho]) = \mathfrak{S}_n\big)$ for IR ρ of $Spin(n)$ are given as $J_\rho(s_i) = c_i\, \nabla^{(2)}(r_i)$ with constants c_i, and $\nabla^{(2)}(z_1) = -I$. So J_ρ is a spin representation of $\mathfrak{S}_n$ and is lifted up to a linear (and spin) representation $\nabla^{(2)}$ of $\widetilde{\mathfrak{S}}_n$.*

(ii) *Suppose $n = 2k \geq 4$ is even. For each of suffices $+$ and $-$, the intertwining operators $J_{\rho_\pm}(s)$ $\big(s \in \mathfrak{S}_n([\rho_\pm]) = \mathfrak{A}_n\big)$ for IR $\rho_\pm$ of $Spin(n)$ are given as $J_{\rho_\pm}(s_i s_j) = c_{ij}\, \mho(r_i r_j)$ with constants c_{ij}, for $i, j \in \boldsymbol{I}_n = \boldsymbol{I}_{2k}$, where*

$$\mho(r_i r_j) := \nabla^{(2)}(r_i)\nabla^{(2)}(r_j) = \frac{1}{2}(Y_i - Y_{i+1})(Y_j - Y_{j+1}). \tag{49}$$

Here, together with $\mho(z_1) = -I$, $\mho$ gives a linear (and spin) representation of the double covering group $\widetilde{\mathfrak{A}}_n := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_n)$ of $\mathfrak{A}_n$, which is generated by $\{r_i r_j;\, i, j \in \boldsymbol{I}_n\}$. So $J_{\rho_\pm}$ is a spin representation of $\mathfrak{A}_n$ and is lifted up to spin IR $J'_{\rho_\pm} = \mho$ of $\widetilde{\mathfrak{A}}_n$.

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