Weyl's dimension formula and higher-order strange formulae of Freudenthal-de Vries on highest weight representations of a semisimple complex Lie algebra

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§1. Historical abstract

In this paper, we give an explicit fourth-order formula, which is a generalization of the second-order strange formula of Freudenthal-de Vries on highest weight representations of a semisimple complex Lie algebra.

There exist several proofs of Weyl's character formula as follows. His original proof leaned on integral formula with respect to Haar measure. The first algebraic proof was due to H. Freudenthal, and then B. Kostant and P. Cartier used multiplicities of weights and recursive formula. A cohomological proof was obtained by I. Bernstein-I. Gelfand-S. Gelfand soon after D. Verma's induced representation theoretical modules.

Applying a systematic procedure of perturbation expansion in Weyl's character formula, we can simultaneously obtain both Weyl's dimension formula, strange formula, and higher-order strange formulae.

In around 1960, so-called strange formula was obtained by Hans Freudenthal while he was elaborating on the algebraic proof of Weyl's dimension formula via Weyl's character formula. We cannot help us feeling strangeness not only for itself but also for its proof. Let us raise a question where dose an idea of the proof come from. In 2012, we happened to come across an article of Hermann Weyl as follows.

"Ramification, old and news, of the eigenvalue problem " , Bull. AMS 56 (1950), 115-139.

To quote Weyl (p.117) ;

when, not so long after, I learned through Felix Bernstein about the problem of mean motion in Lagrange's linear theory of perturbation for the planetary system, ...

This is an example of how experience in one field of mathematics may give one the lead in an entirely different field.

Though I. Newton raised a naive perturbation theoretical idea, the origin of perturbation theory is planetary dynamics in the 18-th century due to L. Euler and J.L. Lagrange.
For example, we could chase approximate orbit of a planet of the solar system by considering the effect of gravity of the sum. In fact, there exist several effects of gravitations of other planets as perturbation forces. Hence, for the sake of effective calculation, we have to consider perturbation effects by introducing a formal parameter (; say t), and some formal power series (with respect to t).

It is well-known since 20-th century that many results in quantum mechanics have been obtained by applying perturbation expansion methods.

Infinite dimensional Lie algebras were begun to investigated by V.G. Kac and R.V. Moody. Especially, Kac obtained a series of " very strange formulae " by using so-called Weyl-Kac denominator formula, theta functions, and modular forms.

§2. Preliminaries and notations

Let G be a complex semisimple Lie algebra with dimension n, and let H be a Cartan subalgebra of G with r = rank G = dim H . Denote by R the root system of G corresponding to H , with a fixed ordering, R_+ and R_- the system of the positive and negative roots, respectively, and S the set of simple roots. Then we have a root space decomposition as follows.

$$G = H \oplus \sum_\alpha G^\alpha$$

Let W be the Weyl group of (G, H) and write by $\epsilon(w) = \det(w)(w \in W)$, and we put $\rho = 1/2 \sum_{\alpha \in R_+} \alpha$. Furthermore, let E be a finite dimensional irreducible representation of G (i.e. E is a G-module) with highest weight $\lambda \in H^* = Hom(H, C)$, and let (β, α) be the bilinear form on H^* induced by the help of the Killing form, Using this form, one can identify $h_\alpha \in$ H and the coroot $\check{\alpha} = 2/(\alpha,\alpha)\ \alpha$ (of α) such that

$$\langle\beta, \check{\alpha}\rangle = \beta (h_\alpha) = 2/(\alpha,\alpha) (\beta , \alpha) \text{ for every root } \alpha \text{ in R.}$$

Write by $E^\mu = \{v \in E | hv = \mu(h)$ v for every h $\in$ H $\} \neq 0, and\ \mu_2 \leq \mu_1 (\mu_1, \mu_2 \in H^*)$ means that

$\mu_1 - \mu_2 \in \{\chi \in H^* | \chi = \sum_{\alpha \in S} n_\alpha \alpha, 0 \leq n_\alpha \in Z\}$. Since $E = \bigoplus_{\mu \leq \lambda} E^\mu$, the character of E (; denoted by ch(E)) can be written as follows.

$ch(E)(\tau) = \sum_{\mu \leq \lambda} (dim E^\mu) exp(\mu, \tau)$, where $\tau \in H^* \cong H$. Here we often use a formal notation ; $\exp(\mu_1 + \mu_2) = exp(\mu_1) exp(\mu_2)$ and and $ch(E) = \sum_{\mu \leq \lambda}$ (dim $E^\mu) exp(\mu)$.

The celebrated character formula of Hermann Weyl says that

$$\left(\sum_w \epsilon(w) exp(w\rho)\right) ch(E) = \sum_w \epsilon(w) exp(w(\lambda + \rho)),$$

$$\sum_w \epsilon(w) exp(w\rho) = \prod_{\alpha \in R_+} \left(exp(\alpha/2) - exp(-\alpha/2)\right).$$

As a corollary, Weyl also obtained the following dimension formula ;

$$\dim E = \prod_{\alpha \in R_+} \langle \lambda + \rho, \check{\alpha} \rangle \langle \rho, \check{\alpha} \rangle^{-1} = \prod_{\alpha \in R_+} (\lambda + \rho, \alpha)(\rho, \alpha)^{-1} .$$

Furthermore, $24(\rho, \rho) = \dim G$ is called Freudenthal-de Vries strange formula.

§3. Main formulae

In this section, we report main result without proof.

(0) (Dimension formula) dim E $= \sum_{\mu \leq \lambda} (dim E^{\mu}) = \prod_{\alpha \in R_+} (\alpha, \lambda + \rho)(\alpha, \rho)^{-1}$.

(1) $\sum_{\mu \leq \lambda} (dim E^{\mu})(\mu, \rho) = 0$.

(2) (second-order strange formula) $\sum_{\mu \leq \lambda} (dim E^{\mu}) 1/2 (\mu, \rho)^2 = (dim E)/48 (\lambda, 2\rho)$.

(2m+1)(m=0, 1, 2, ...)(odd-order vanishing formulae) $\sum_{\mu \leq \lambda} (dim E^{\mu})(\mu, \rho)^{2m\text{-}}$
0.

(4)(fourth-order strange formula) $1/4! \sum_{\mu \leq \lambda} (dim E^{\mu})(\mu, \rho)^4$

$= (\text{ dim E })/4608 \{(\lambda + \rho, \lambda + \rho)^2 + (\rho, \rho)^2\} - (dim E)/2880 \{\sum_{\alpha \in R_+} (\alpha, \lambda)^4$

$+ 4 \sum_{\alpha \in R_+} (\alpha, \lambda)^3 (\alpha, \rho) + 6 \sum_{\alpha \in R_+} (\alpha, \lambda)^2 (\alpha, \rho)^2 + 4 \sum_{\alpha \in R_+} (\alpha, \lambda)(\alpha, \rho)^3 \}$.

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