H. Poincaré's contribution to probability theory

Si Si
Faculty of Information Science and Technology
Aichi Prefectural University

1 Introduction

H. Poincaré 1851-1912

This year is the 100th anniversary of Professor H. Poincaré's death. We wish to pay our respect to his memory.

There are many publications by him on probability theory. One remarkable fact is that his idea on probability seems to be the same as J. Bernoulli. In fact, Poincaré is some years younger than Bernoulli.

Poincaré might have been influenced by I. Kant, Kritik der Reinen Vernunft (1787). We can see it in Part I, 1. Von dem Raume 2. Von der Zeit.

Namely, Poincaré claims that gamble is not referred to@as the origin of probability theory. He is absolutely right. With this fact in mind, we shall briefly study his assertion related to probability theory.

He did important work in many different branches of mathematics. However, he did not stay in any one field long enough to round out his work. He had an amazing memory and could state the page and line of any item in a text he had read. He kept this memory all his life, but not in the manner of learning by rote, rather by linking the ideas he was assimilating particularly in a visual way. He also remembered in exactly the same words by ear. He normally solve a problem completely in his head, then carry out the completed problem to be a paper. He was also a popular mathematician.

2 Brief discussion on his works

In 1862 Henri entered the Lycee in Nancy (now it is called the Lycee Henri Poincaré in his honour). He spent eleven years at the Lycee and during this time he was one of the top students in every topic he studied. Henri was described by his mathematics teacher as a "monster of mathematics" and he won first prizes in the concours general, a competition among the top pupils from all the Lycees across France.

Poincaré entered the Ecole Polytechnique in 1873, graduated in 1875.

Poincaré began with popular science writings and progressing to more advanced texts. Poincaré continued his studies at the Ecole des Mines, after graduated from the Ecole Polytechnique, .

As a student of Charles Hermite, Poincaré obtained his doctorate in mathematics from the University of Paris in 1879. His thesis was on differential equations.

Poincaré was appointed at the University of Caen for teaching mathematical analysis after getting his doctorate.

In 1886 Poincaré was nominated for the chair of mathematical physics and probability at the Sorbonne with the intervention and the support of Hermite. He was also appointed to a chair at the Ecole Polytechnique. Poincaré held these chairs in Paris until his death at his early age of 58.

In Paris he changes his lectures every year, and reviewed optics, electricity, the equilibrium of fluid masses, the mathematics of electricity, astronomy, thermodynamics, light, and probability.

Poincaré was a scientist taken up by many aspects of mathematics, physics and philosophy, and he is often described as the last universalist in mathematics. He contributed to numerous branches of mathematics, celestial mechanics, fluid mechanics, the special theory of relativity and the philosophy of science. Much of his research involved interactions between different mathematical topics and his broad understanding and knowledge allowed him to attack problems from many different angles.

Before the age of 30 he developed the concept of automorphic functions which are functions of one complex variable invariant under a group of transformations characterized algebraically by ratios of linear terms. The idea was to come in an indirect way from the work of his doctoral thesis on differential equations. It was said that the crucial idea came to him as he was about to get onto a bus, as he relates in Science and Method (1908).

Poincare's book "Analysis situs" , published in 1895, is an early systematic way of dealing topology. We can think him as the originator of algebraic topology. In 1901, he claimed that his researches in many different areas such as differential equations and multiple integrals had all led him to topology. Poincaré published the first of his six papers on algebraic topology in 1894, essentially all of the ideas and techniques in the subject were based on his work. Even today the Poincaré conjecture remains as one of the most baffling and challenging unsolved problems in algebraic topology.

Poincaré is also considered as the originator of the theory of analytic functions of several complex variables. He began his contributions to this topic in 1883 with a paper in which he used the Dirichlet principle to prove that a meromorphic function of two complex variables is a quotient of two entire functions. He also contributed in algebraic geometry, maybe fundamental aspects, writing papers in 1910-11.

He also wrote many popular scientific articles at a time when science was not so popular among the general public in France. After Poincaré became a prominent mathematician, he turned writing to describe the meaning and importance of science and mathematics for the general public.

Poincaré's popular works include Science and Hypothesis (1901), The Value of Science (1905), and Science and Method (1908). A quote from these writings is particularly relevant to this archive on the history of mathematics. In 1908 he wrote:-

The true method of foreseeing the future of mathematics is to study its history and its actual state.

3 Poincaré's contributions to the philosophy of mathematics and science

First, we mention that Poincaré viewed logic and intuition as playing a part in mathematical discovery. He wrote in Mathematical definitions in education (1904):-

It is by logic we prove, it is by intuition that we invent.

In a later article Poincaré emphasised the point again in the following way:-

Logic, therefore, remains barren unless fertilised by intuition.

Poincaré achieved the highest honors for his contributions of true genius. He was elected to the Academie des Sciences in 1887 and in 1906 was elected President of the Academy. The breadth of his research led to him being the only member elected to every one of the five sections of the Academy, namely geometry, mechanics, physics, geography and navigation sections. He was also made chevalier of the Legion d'Honneur and was honored by a large number of learned societies around the world. He won numerous prizes, medals and awards.

In 1908 he was elected to the Academie Francaise and was elected director in the year of his death.

A quotation from an address at the funeral:-

[M Poincare was] a mathematician, geometer, philosopher, and man of letters, who was a kind of poet of the infinite, a kind of bard of science. Article by: J J O'Connor and E F Robertson

Honours awarded to Henri Poincare are
LMS Honorary Member 1892
Fellow of the Royal Society 1894
Fellow of the Royal Society of Edinburgh 1895
Speaker at International Congress 1897
Speaker at International Congress 1900
Royal Society Sylvester Medal 1901
Speaker at International Congress 1908
ASP Bruce Medallist 1911
Paris street names : Rue Henri Poincare (20th Arrondissement)

4 Poincaré's work on Probability

As is known Poincaré is one of the greatest philosophers of Science. Among Poincaré's work *Probability* is just minor among his works. However, there was a book written by him,

gCalcul des Probabilités, Gauthier-Villar ,Paris, (1896).
Second edition. 1912, the year he died. He did not present the recent advances in his book.

This book is a pivotal text, the first one is written by J. Bertrand,
gCalcul des Probabilités s Gauthier-Villar ,Paris, (1889).

Poincare held the chair of Probability and Mathematical Physics at the Faculé des Sciences de Paris. He gave a new course almost every year, and most of them were written up and published. Probability is one of them.

Between 1896 and 1912, involvement of Probability in Physics can be seen ;the work of Gibbs, completing that of Maxwell and Boltzmann; Research of Plank in thermodynamics, and the law of black-body radiation (1900).

In the beginning of 20th century there was a sudden change in the calculus of probability. Louis Bachlier's Thesis 1900 (H. Poincare is a reporting jury member) began a long line works on Brownian motion unnoticed , no influence on the theory s development.

Concerned with Brownian motion, there are works of
Paul Lévy : Probability law on abstract sets, 1911,
Nobert Wiener : Differential space 1923,
Paul Lévy : Calcul de probabilities 1925.

Wiener refers Levy's book Lecons d'analyse fonctionnelle 1922 in his paper "Differential space" (1923). But in reality he uses the result on white noise (that is Brownian motion) of the Lévy paper "Probability law on abstract sets" published in 1911 in the journal "Revue de Me'taphysique et de morales. In this paper Lévy showed that the ideal measure (like Lebegue measure) on an infinite dimensional space (does not mean a Hilbert space, but a space where all the coordinates play the same role uniformly) is white noise, in his sense, a uniform measure on the infinite dimensional sphere with radius $\sqrt{\infty}$. This is an intuitive understanding the white noise measure.

Wiener says "differential" space. This means that he is taking difference of Brownian motion. The limit is white noise.

In this period, the most profound works are done by E. Borel, Fréchet and others

In 1905 Borel applied his measure theory and the Lebesgue integral to the theory of probabilities in his paper Remarks on some questions of probability. In 1906, Maxwell-Boltzman law is proved. In 1909, he dealed with the countable probabilites which can be seen in "Les probabilities denombrables et leuers applications arthmetiques cc".

Fréchet discussed the "Abstract formulation of measure theory which leads to Kolomogorov's axiom system" in 1915 .

Poincaré has not mentioned measure theory. For a function (which will serve as density function) he assumes it to be continuous. He often did like a physicist, did not take responsibility about integration by parts or expansion into a convergent Taylor series. He did not extinguish between Fourier transform (always exists) and Lapace transform (which assumes strong hypotheses on the moments of a random variable).

5 Poincaré's valuable hidden conceptual analysis in his Probability book

Introduction to the 1912 is the reprint of 1905 article.It analyzes the notion of chance and the possibility of glaws of chanceh
First of all, he takes the viewpoint of objective probabilities and refuses to reduce chance to simple ignorance; this is the incursion, made by a careful scholar, into the ideological realm. He gave *two main Characteristics of chance.*

First characteristic of chance is mentioned in his book :

gA very small cause, which escapes us, determines a significant effect that we cannot miss, and then we say that this effect due to chance c.it can happen that small differences in the initial conditions generate very large ones in the final phenomena c.Prediction becomes impossible and we have the phenomenon of accident.h

The important philosophical point is that determinism is not contradicted. Chance does not at all proceed from ignorance, but from an insufficient precision in the data.

it The second characteristic of chance comes from the frequency. We will omit this part.

The mechanism of chance is mentioned as

gcso that we can predict, if not what are the effects in each case, but at least what these effects will be on average c..h

For *Illumination of the exponential principle*, he said that

gcif the first collision multiplied the deviation a very large number A, after n collisions it w ill be multiplied by A^n which will be very large, not only for large A but also for the exponent large n.

This would become the Lyapunov stability criterion and would be the basis for the definition of stable and unstable dynamical system. And we see the obvious importance of eigenvalues, that is, multipliers of such as the above A.

It can be seen in the following example.

There are 2 states A and B. At each step, the probability of remaining at the present state is p_1 and the probability of transition from one state to another is p_2.

One dollar bet for returning to the initial state after n steps. The mathematical expectation of the gain will be $G_n = (p_1 - p_2)^n$. If p_1 or p_2 is not too close to 1 the gain will be small for large n, so the game is fair regardless of the values of p_1 and p_2 (Poincare's conclusion). This is one of the main manifestation of ergodic theorems for Markov chains.

Laplace gave the definition of the probability as the proportion of number of favorable cases and number of possible cases with the assumption of equiprobable.

Modern definition is given as follows:

Ω is the set of possible events

Each element ω_i is associated to $p_i \geq 0, \sum p_i = 1$

For a subset A of Ω, it s probability is defined as

$$P(A) = \sum_{\omega_i \in A} p_i$$

If p_i are all equal, Lapalace definition is covered. If p_i are rational we can reduce the Laplace definition.

Regarding to Poincaré, if p_i are irrational only appears when passing a limit (e.g. a game played with a very large number of times).

Continuous probabilities

Although Poincaré recongnized the trap and the paradoxes discovered by his predecessors, he did not completely innovate about the continuous probabilities, but his analysis made a progressive liberation from the distrust of continuous probabilities.

Modern version of the definition can be expressed as

1. For a given experiment, Ω is the corresponding sample space.
2. Ω is a priori an abstract measure space; but in most natural applications, it can be reduced to work on a separable complete metric space and then the notion of a Borel set can be defined.
3. To each Borel set B of Ω, its probability $P(B) \geq 0$ is associated; it satisfies the countably additive law and $P(\Omega) = 1$.

Thus, the mathematical model is characterized unambigously but it is difficult to construct the probability measure on Ω. Due to Kolmogorov, Prokhorov, Minlos, Shinai, there are now powerful analytic techniques.

The preceeding analytic methods came much later than Poincaré, and would require the Lebesgue integration, but his conceptual analysis is correct. Now this method is quite familiar.

Probabilistic model

1 The probabilistic mathematical model is defined by choice.@

2 The choice of ω is arbitrary, but after chosen it must be kept.

3 In case ω is concretised, it must satisfy certain principles of symmetry which are expressed in the form of the invariance of ω under a group G of transformations of the configuration space.

In certain case, the above condition 3 is enough to determine ω which are the interesting cases of *Geometric probability.* The most important point for Poincaré is that in certain situation, the final result is independent of the law ω chosen priori since Poincaré is much interested in connection with physics; while we now consider that ω determine randomness.

Gauss's law

At the end of the 19th century, people work on the probability discussed on the impact of Gauss's law. The analytic investigation on this subject led to various formulation of Central limit theorem. Poincaré discusses also in his book, but it is far from modern analytic rigor.

P. Lévy pointed out that Poincaré introduced for the first time the notion of characteristic function, written in the form

$$f(\alpha) = \sum p e^{\alpha x}$$

$$f(\alpha) = \int \varphi(x) e^{\alpha x} dx$$

Now it is written as $f(\alpha) = E[e^{\alpha x}] = \Phi_X(\alpha)$.

It seems that he used α since Laplace transform always exists for complex α.

If we take $i\alpha$ instead of real α, (i.e. Fourier transform) then we have the characteristic function. Then the central limit theorem can be formulated as follows.

Let X be a random variable with mean m and finite variance σ^2. We take a sample $X_1, X_2, \cdots, X_N$ of N independent identically distributed random variables with the same distribution of X. Let Θ_N be a normalized random error such that

$$X_1 + X_2 + \cdots + X_N = mN + \sigma\sqrt{N}\Theta_N.$$

Then for large N, Θ_N it approximately follows the Gauss s law with density

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{1}$$

Assuming mean 0 and variance 1, the characteristic function Φ is of class C^2, with

$$\Phi(0) = 1, \Phi'(0) = 0, \Phi''(0) = 1 \tag{2}$$

The characteristic function of Θ_N is $\phi(\sqrt{N}\alpha)^N$ and then

$$\lim_{N\to\infty} \phi(\sqrt{N}\alpha)^N = e^{-\alpha^2/2} \tag{3}$$

which corresponds to the density (1). This is essentially reasoned by Poincaré, he uses the fact that all the moments $E[X^n]$ are finite and takes expansions into Taylor series, even though a finite expansion suffices to pass from (2) to (3).

Here we must specify what it means that approximately following a probability law. One way is to introduce a notion of convergence in the set of probability measures on R, the vague topology (defined by Henri Cartan in 1941 in a more general frame work).

It must be also proved the continuity of the Fourier transform.

$$\mu \to \phi(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} d\mu(x),$$

where α is real. It transforms the vague topology of probability measure to the convergence of characteristic functions. This is the classical theorem due to P. Lévy, and was proved by him in 1922 and explained in his 1925 book.

Conclusion

Poincaré's book on probability is neither deep and nor having the novelty of Analysis Situs (by Poincaré 1986). However, it is still a 19th century work but it is one of the first work entered into the games of statistical physics. In this sense this book already belongs to the 20th century paving the way of Einstein, Wiener and others.

References

[1] H. Poincaré, L'analyse et la recherche.

Reflexions sur le calcul des probabilités

Revue Générale des Sciences Pures et Appliquées,1901,994-1007.

Collection savoire: Sciences. Hermann.

[2] H. Poincaré,Science et l'hypothesis. 1902.

[3] H. Poincaré, Calcul des probabilité. 1912.

[4] The Scientific legacy of Poincaré, ed. Eric Charpentier et.al. American Mathematical Soceity.2010.