調和写像と2-調和写像の歴史と現況

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２０１１年１０月２９日

調和写像と2-調和写像の歴史の概観

- 調和写像 の研究は
  J. Eells and J. H. Sampson, 1964 により始まった.
  (cf. HARMONIC MAPS, Selected Papers of James Eells and Collaborators, World Scientific, 1992)
  に J. Eells の仕事がまとめられている.
- 他方 2-調和写像の研究は
  G. Y. Jiang, in 1986. (cf. G.Y. Jiang, *2-harmonic maps and their first and second variational formula*, Chinese Ann. Math., 7A (1986), 388–402) に始まる.
- G. Y. Jaing (姜国英) の論文は中国語で書かれていた. このため, 彼の仕事は理解されないでいた.

姜国英と論文の英語訳

- 数奇な運命をたどり, 英語訳 は, H. Urakawa により英訳され, 出版された：
  G.Y. Jiang, *2-harmonic maps and their first and second variational formula, Note di Matematica*, 28 (2009), 209–232.
- 今日の話は, その経緯についてお話ししたい.
- その後, 調和写像や 2-調和写像とは, どのような概念なのか, 簡単にお話しする.

2-調和写像との出会い（1）

- 白状すれば, 5年前まで, 2-調和写像について何も知らなかった.
- 日本人で, 2-調和写像の最初の専門家は,
  石川　晋（いしかわ　すすむ）氏,
  当時,　佐賀大学教育学部,
  現在, 福岡工業大学情報工学部,
  のようである.
- B. Y. Chen & S. Ishikawa,
  Biharmonic surfaces in pseudo-Euclidean spaces.
  Mem. Fac. Sci. Kyushu Univ. Ser. A 45 (1991), no. 2, 323–347.

2-調和写像との出会い（2）

- ２００６年２月９日, Workshop 「Differential Geometry, Sendai 2006」が東北大学青葉記念会館７階で開催（西川青季先生主催）
- Eric Loubeau (Universite de Brest, France) の講演:『Biharmonic maps』講演中身は忘却, ノートのみ.
- 井ノ口順一氏（宇都宮大学教育学部, 現在山形大学理学部）より依頼（２００７年１月１０日）:
  姜国英, 2-調和映照及基第一, 二変分公式,
  数学年刊, 7 A (4) (1986), 388 – 402.
  『雑誌「数学年刊」が東北大学理学部と早稲田大学理工学部にだけあるので, コピーしてほしい』
- コピーついでに読み始めたというのが, きっかけ.

姜国英, 2-調和映照及基第一, 二変分公式

- ２００７年１月１０日より姜論文（中国語）を読み始める.
- いくつかの式が印刷ミスで大きく崩れている.
- ２００７年１月２６日, 姜論文（中国語）が証明を補うことができ, 大筋で正しいことを確認.
- ２００７年１月２０日, 姜論文の「ヤング・ミルズ場」アナロジーができる.
- ２００７年１月２１日, Cheng 予想の一般化.
- ２００７年１月２７日, 姜論文の仕事を「複素射影空間, 四元数射影空間での類比」の研究開始.
- ２００７年２月１４日, 球面内の 2-調和等径超曲面の分類完成（姜論文の拡張）.

研究会で

- ２００７年６月１３日～６月１６日，イタリア，レッチェでの研究会：The international conference “Advances in Differential Geometry” in honor of Prof. Oldrich Kowalski,
  ２ - 調和写像のこれまでの研究をまとめて発表，姜論文の英訳についても言及。
- 研究会後，我々の研究と姜論文英訳を依頼される.
- 姜国英氏（数学研究を止めて，渡米）と連絡取れず.
- Editor-in-chief of Proc. of the conference より Editor-in-chief of Chinese Ann. Math. に英訳承諾を得て，浦川 肇訳掲載: ４カ所，証明を長文補足.

出版される

- 出版される:
  - Jiang Guoying, *2- harmonic maps and their first and second variational formulas*, Note di Matematica, **28**, suppl. n. 1, (2009), 209–232.
  - T. Ichiyama, J. Inoguchi and H. Urakawa, *Bi-harmonic maps and bi-Yang-Mills fields*, Note di Matematica, **28**, suppl. n. 1, (2009), 233–275.
- ２０１０年７月姜国英よりメール：私は学位取得後渡米し，Ann Arbor 米国数学会 Math. Reviews で働いている．友人から最近，「私の名の論文が出版されている」と注意されて，見た．大変有難う．８月復旦大学に行く．私の学位論文を進呈します．

学位論文を進呈される

- ２０１０年８月３１日，学位論文を進呈される：
- 復旦大学研究生論文：
  Riemann 流形間的２重調和映照与守恒律
  系研究所：数学研究所,
  專業：基礎数学,
  研究方向：微分幾何,
  姓名：姜国英,
  申請学位：理学博士,
  指導教師：蘇歩青，胡和生,
  完成日期：１９８４年７月

その後

- 浦川　肇：「私の数学感覚」
- 数理科学，２０１２年１月号，No. 583 で出版予定：
- 6. １節：k - 調和関数とその特徴付け定理
  6. ２節：調和写像と k - 調和写像
- M. Nicolesco, *Recherche sur les fonctions polyharmoniques*, Ann. scient. Ecole Norm. Sup., **52** (1935), 183–220.
  M. Nicolesco, *Les fonctions polyharmoniques*, Hermann, Paris, 1936.
  をご覧下さい。

ユークリッド空間上の調和関数

- $\Omega \subset \mathbb{R}^m$ を m 次元ユークリッド空間内の開領域とする.
- Ω 上の調和関数 $f(x)$ とは

$$\Delta f = 0 \quad (\Omega \text{ 上})$$

が成り立つことである. ラプラシアン Δ は

$$\Delta := \sum_{i=1}^{m} \frac{\partial^2}{\partial {x_i}^2}$$

である. ただし $\mathbb{R}^m$ の座標を, $x = (x_1, \cdots, x_m)$ と表す.

k - 調和関数

- Ω 上の k - 調和関数 $f(x)$ とは

$$\Delta^k f = \Delta(\cdots(\Delta f)\cdots) = 0 \quad (\Omega \text{ 上})$$

となることである.
- $k = 2$ のとき, 2 - 調和関数は重調和関数とも呼ばれている.
- Ω が原点中心の星形領域のとき k - 調和関数の次のようなアルマンシ表現定理が知られている.
  Ω が原点中心の星形領域とは,
  「$x \in \Omega$ かつ $0 \leq \alpha \leq 1$ ならば, $\alpha x \in \Omega$ となる」ことである.

アルマンシ表現定理

アルマンシ表現定理　Ω は m 次元ユークリッド空間内の原点中心の星形領域とする. このとき,
(1) Ω 上の k - 調和関数 $f(x)$ は次のように表示される：

$$f(x) = |x|^{2k-2}h_k(x) + \cdots + |x|^2 h_2(x) + h_1(x)$$

ここで $|x|^2 = \sum_{i=1}^m x_i^2$ であり, h_i $(i = 1, \cdots, k)$ は Ω 上の k 個の調和関数である.
(2) $f(x)$ が完全優調和関数とは, すべての $j = 1, \cdots, k$ について,

$$(-\Delta)^j f(x) \geq 0 \quad (\Omega \text{ 上})$$

となることで, このときの $h_k, \cdots, h_1$ は交互に正値関数, 負値関数となって現れる.

Some unsolved problems

- Assume that (M, g) is a complete Riemannian manifold. Can every k-harmonic function $f(x)$ on (M, g) be expressed as

$$f(x) = r(x)^{2k-2}h_k(x) + \cdots + r(x)^2 h_2(x) + h_1(x)$$

  in terms of harmonic functions h_i $(i = 1, \cdots, k)$ on (M, g) ?　Here, $r(x) = d(x, x_0)$ $(x \in M)$ is the distance function from some fixed point $x_0 \in M$.
- Can one extend the above theorem to k-harmonic maps ?
- Namely, can any k-harmonic map be described in terms of harmonic maps ?

Robotics and harmonic maps

- P.C. Park & R.W. Brockett, Kinematic dexterity of robotic mechanisms, *Intern. J. Robotics Res.*, **13** (1994), 1–15.
  Y.J. Dai, M. Shoji & H. Urakawa, Harmonic maps into Lie groups and homogeneous spaces, *Differ. Geom. Appl.* **7** (1997), 143–160.
-

work space	target space
revolute, prismatic joint	Tori, Euclidean space
kinematic distortion	energy

- Which kinematic design results in minimum kinematic distortion ?

From the submanifold theory (1)

- B.Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.*, **17** (1991), 169–188.
- Consider an isometric immersion $f : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ and $f(x) = (f_1(x), \cdots, f_k(x))$ $(x \in M)$. Then,
- $\Delta f := (\Delta f_1, \cdots, \Delta f_k) = -m\,\mathbf{H}$,
- $\mathbf{H} := \frac{1}{m}\sum_{i=1}^m B(e_i, e_i)$, the mean curvature vector field,
  $B(X, Y) := D^0_X(f_*Y) - f_*(\nabla_X Y)$, the second fundamental form.

From the submanifold theory (2)

- Def $f : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ is minimal if $\mathbf{H} \equiv 0$.
- Chen defined that f is biharmonic if

$$\Delta\mathbf{H} = \Delta(\Delta f) \equiv 0.$$

- Thm (Chen) If $\dim M = 2$, any biharmonic submanifold is minimal.
- B.Y. Chen's Conjecture:
  Any biharmonic isometric immersion into $(\mathbb{R}^k, g_0)$ is always minimal.

Definitions

- For a smooth map $f : (M, g) \to (N, h)$, the energy functional is:　$E(f) := \frac{1}{2}\int_M \|df\|^2 v_g$.
- The first variation formula is:

$$\left.\frac{d}{dt}\right|_{t=0} E(f_t) = -\int_M \langle \tau(f), V\rangle v_g = 0,$$

- where

$$\tau(f) := \sum_{i=1}^m B(f)(e_i, e_i),$$

$$B(f) := \nabla^N_{df(X)} df(Y) - df(\nabla_X Y).$$

- $f : (M, g) \to (N, h)$ is harmonic if $\tau(f) = 0$.

The second variation formula

- The second variation formula for the energy functional $E(f)$ is

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(f_t) = \int_M \langle J(V), V\rangle v_g,$$

- where

$$J(V) := \overline{\Delta} V - \mathcal{R}(V),$$

$$\overline{\Delta} V := \overline{\nabla}^* \overline{\nabla} V, \quad \mathcal{R}(V) := \sum_{i=1}^{m} R^N(V, df(e_i))df(e_i).$$

Poly-harmonic maps

- The k-energy functional due to Eells-Lemaire is

$$E_k(f) := \frac{1}{2}\int_M \|(d+\delta)^k f\|^2 v_g \ (k = 1, 2, \cdots),$$

where it turns out that $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g$.
- The first variation formula for $E_2(f)$ is given by

$$\frac{d}{dt}\bigg|_{t=0} E_2(f_t) = -\int_M \langle \tau_2(f), V\rangle v_g,$$

$$\tau_2(f) := J(\tau(f)) = \overline{\Delta}\tau(f) - \mathcal{R}(\tau(f)).$$

- $f : (M, g) \to (N, h)$ is biharmonic if $\tau_2(f) = 0$.

The first variation of poly-harmonic maps

- The first variation of $E_k(f)$ is given (cf. IIU), as

$$\frac{d}{dt}\bigg|_{t=0} E_k(f_t) = -\int_M \langle \tau_k(f), V\rangle v_g,$$

where

$$\tau_k(f) := J(W_f) = \overline{\Delta}(W_f) - \mathcal{R}(W_f),$$

$$W_f := \overline{\Delta}\cdots\overline{\Delta}\,\tau(f) \in \Gamma(f^{-1}TN).$$

- $f : (M, g) \to (N, h)$ is k-harmonic if $\tau_k(f) = 0$.

Second variation formula (due to Jiang)

The second variation formula for $E_2(f)$ is given by

$$\frac{d^2}{dt^2}\bigg|_{t=0} E_2(f_t) = \int_M \langle J_2(V), V\rangle v_g,$$

$$J_2(V) = J(J(V)) - \mathcal{R}_2(V),$$

$$\begin{aligned}\mathcal{R}_2(V) = &\ R^N(\tau(f), V)\tau(f)\\ &+ 2\,\mathrm{tr}R^N(df(\cdot), \tau(f))\overline{\nabla}.V + 2\,\mathrm{tr}R^N(df(\cdot), V)\overline{\nabla}.\tau(f)\\ &+ \mathrm{tr}(\nabla^N_{df(\cdot)}R^N)(df(\cdot), \tau(f))V\\ &+ \mathrm{tr}(\nabla_{\tau(f)}R^N)(df(\cdot), V)df(\cdot).\end{aligned}$$

Indices and nullities

- The index and nullity for a harmonic map are defined by

$$\mathrm{Index}(f) := \dim(\oplus_{\lambda<0}E_\lambda), \mathrm{Nullity}(f) := \dim E_0.$$

- The index and nullity for a biharmonic map are defined by

$$\mathrm{Index}_2(f) := \dim(\oplus_{\lambda<0}E^2_\lambda), \mathrm{Nullity}_2(f) := \dim E^2_0.$$

- Here E_λ, E^2_λ are the eigenspace of J, J_2 with the eigenvalue λ, respectively.
- Thm If f is a harmonic map, it is biharmonic and

$$\mathrm{Index}_2(f) = 0, \mathrm{Nullity}_2(f) = \mathrm{Nullity}(f).$$

Biharmonic maps into S^n

- Thm (Jiang) Let $f : (M^m, g) \to S^{m+1}(\frac{1}{\sqrt{c}})$ be an isometric immersion. Assume that the mean curvature of f is nonzero constant. Then, f is biharmonic if and only if $\|B(f)\|^2 = mc$.

- Using this theorem, we give a classification of biharmonic isoparametric hypersurfaces in $S^n(1)$.

Isoparametric hypersurfaces in S^n

- Let $f : (M, g) \to S^n(1)$ be an isometric immersion, and $\dim M = n - 1$.
- Let us recall the shape operator $A_\xi : T_xM \to T_xM$ ($x \in M$) which is

$$g(A_\xi X, Y) = \langle f_*(\nabla_X Y), \xi \rangle, \quad X, Y \in \mathfrak{X}(M),$$

  where ξ is the unit normal vector field along M.
- The eigenvalues of A_ξ are called the principal curvatures. M is called isoparametric if all the principal curvatures are constant in $x \in M$.

Cartan, Münzner, Ozeki, Takeuchi

- Assume that $f : (M, g) \to S^n(1)$ is an isoparametric hypersurface. Then, there exists a homogeneous polynomial F on $\mathbb{R}^{n+1}$ of degree d such that M is given by

$$M = f^{-1}(t), \text{ for some } -1 < t < 1,$$

  where $f := F|_{S^n(1)}$. Say $M = M(t)$.
- All the principal curvatures are given as

$$k_1(t) > k_2(t) > \cdots > k_{d(t)}(t),$$

  with their multiplicities $m_j(t)$ ($j = 1, \cdots, d(t)$).
- $d = d(t)$ is constant in t, and $d = 1, 2, 3, 4, 6$.

Classification of biharmonic isopara. in S^n

- Thm Let $f : (M, g) \to S^n(1)$ be a biharmonic isoparametric hypersurface in the unit sphere. Then, (M, g) is one of the following three cases:
- $M = S^{n-1}(\frac{1}{\sqrt{2}}) \subset S^n(1)$ (a small sphere, Oniciuc),
- $M = S^{n-p}(\frac{1}{\sqrt{2}}) \times S^{p-1}(\frac{1}{\sqrt{2}}) \subset S^n(1)$ (the Clifford torus, Jiang), $n - p \neq p - 1$,
- $f : (M, g) \to S^n(1)$ is minimal.

Biharmonic maps into $\mathbb{C}P^n$

- Thm Let (M, g) be a real $(2n - 1)$-dim. compact Riemannian manifold, $f : (M, g) \to \mathbb{C}P^n(c)$, an isometric immersion into the projective space with constant holomorphic sectional curvature c.
- Assume that $f : (M, g) \to \mathbb{C}P^n(c)$ has nonzero constant mean curvature. Then,
- f is biharmonic if and only if $\|B(f)\|^2 = \frac{n+1}{2}c$.

Homogeneous real hypersurfaces in $\mathbb{C}P^n$

- Let us recall classification of all real homog. hypersurf. in $\mathbb{C}P^n$ due to R. Takagi, 1973.
- Let U/K be a Hermitian symmetric space of rank two, and let $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomp.
- $\hat{M} = \mathrm{Ad}(K)A \subset \mathfrak{p}$ is a hypersurface in S^{2n+1} for some regular element $A \in \mathfrak{p}$ with $\|A\| = 1$. Here, we put $\dim_{\mathbb{C}} \mathfrak{p} = n + 1$.
- $M = \pi(\hat{M}) \subset \mathbb{C}P^n$ give all real homogeneous hypersurfaces in $\mathbb{C}P^n$, where

$$\pi : \mathbb{C}^{n+1} - \{0\} = \mathfrak{p} - \{0\} \to \mathbb{C}P^n$$

  is the natural projection.

All homogeneous real hypersurfaces in $\mathbb{C}P^n$

- All homogeneous real hypersurfaces in $\mathbb{C}P^n$ are classified into the following five types (R. Takagi, 1973):
- (A type) $U/K = \frac{SU(s+1)\times SU(t+1)}{S(U(s)\times U(1))\times S(U(t)\times U(1))}$,
- (B type) $U/K = SO(m+2)/(SO(m) \times SO(2))$,
- (C type) $U/K = SU(m+2)/S(U(m) \times U(2))$,
- (D type) $U/K = O(10)/U(5)$,
- (E type) $U/K = E_6/(\mathrm{Spin}(10) \times U(1))$.

All biharm. homog. real hypersurf. in $\mathbb{C}P^n(4)$

- Thm
  Let M be a homog. real hypersurface in $\mathbb{C}P^n(4)$, so that M is one of the types $A \sim E$.
- (I) For all the types, there exists a unique orbit M which is a minimal hypersurface in $\mathbb{C}P^n(4)$.
- (II) There exists a unique orbit $M \subset \mathbb{C}P^n(4)$ which is biharmonic but not harmonic in each the types A, D and E.
  There are no such orbits in the types B, C.

Biharmonic hypersurfaces in $\mathbb{H}P^n(c)$

- Thm Let $\varphi : (M, g) \to \mathbb{H}P^n(c)$ be an isometric immersion with nonzero constant mean curvature, $\dim M = 4n - 1$. Then,
  φ: biharmonic $\Longleftrightarrow$ $\|B(\varphi)\|^2 = (n+2)c$.
- For the cases of non-compact duals ($c < 0$), it holds that

$$\|B(\varphi)\|^2 = mc,\ \frac{n+1}{2}c, \text{ or } (n+2)c \text{ (resp.)}.$$

I.e., any biharmonic hypersurfaces in $(\mathbb{R}^n, g_0)$, or one of the classical rank one symmetric spaces of non-compact type with constant mean curvature is minimal.

Classification of all biharmonic homogeneous hypersurf. in $\mathbb{H}P^n(4)$

- Thm
- (I) (J. Berndt) All the homogeneous real hypersurfacecs in $\mathbb{H}P^n(4)$ are classified into three types.
- (II) In each types, there exist minimal homogeneous real hypersurfaces in $\mathbb{H}P^n(4)$.
- (III) In each types, there exist biharmonic nonminimal homogeneous real hypersurfaces in $\mathbb{H}P^n(4)$.

Chen, Caddeo, Montaldo, Piu and Oniciuc's conjecture

- (B.Y. Chen's conjecture)
  Any biharmonic submanifold of the Euclidean space is harmonic.
- (Caddeo, Montaldo, Piu and Oniciuc's conjecture)
  Any biharmonic immersion into a complete Riemannian manifold with nonpositive curvature is harmonic.
- Solved negatively by Y. Ou=L. Tang, at 2010.6.9:
  ∃ proper biharmonic hypersurfaces into the 5-dim. conformally flat space with strictly negative sectional curvature (arXiv:1006.1838).

Our answer to the conjecture

Thm Assume that (M, g) and (N, h) satisfies $|\mathrm{Riem}^M| \leq C$, and $\mathrm{Riem}^N \leq 0$. Let $f : (M, g) \to (N, h)$ be a biharmonic map whose tension field $\tau(f)$ satisfies

$$\|\tau(f)\| \in L^2(M) \text{ and } \|\overline{\nabla}\tau(f)\| \in L^2(M).$$

Then, $f : (M, g) \to (N, h)$ is harmonic.

All the above results are due to the joint work with T. Ichiyama and J. Inoguchi:

T. Ichiyama, J. Inoguchi and H. Urakawa,
(1) Biharmonic maps and bi-Yang-Mills fields, Note di Matematica, **28** (2009), 233–275.
(2) Classification and isolation phenomena of biharmonic maps and bi-Yang-Mills fields, Note di Matematica, 2009.
ArXiv: 0912.4806.

Conformal change and biharmonic maps

- Let us recall:
  P. Baird & D. Kamissoko, On constructing biharmonic maps and metrics, Ann. Global Anal. Geom., **23** (2003), 65–75.
- Our setting is a little bit different from them:

  Consider a C^∞ mapping $\varphi : (M, \overline{g}) \to (N, h)$ with $\overline{g} = f^{2/(m-2)} g$, $f \in C^\infty(M)$, $f > 0$. $(m := \dim M > 2)$

The identity map of the Euclidean space

- Let $(M, g) = (\mathbb{R}^m, g_0)$, $(m \geq 3)$, the standard Euclidean space, and $f \in C^\infty(\mathbb{R}^m)$ is given by

$$f(x_1, x_2, \cdots, x_m) = f(x_1) = f(x).$$

- Then, $\mathrm{id} : (\mathbb{R}^m, f^{2/(m-2)} g_0) \to (\mathbb{R}^m, g_0)$ is biharmonic

$$\Longleftrightarrow f^2 f''' - 2\frac{m+1}{m-2} f f' f'' + \frac{m^2}{(m-2)^2} f'^3 = 0.$$

Our theorems (Joint work with H. Naito)

- Thm Assume that $m \geq 3$. Then,
- (i) $(m \geq 5)$ There exists no positive global C^∞ solution f on $\mathbb{R}$ of the ODE.
- (ii) $(m = 4)$ $f(x_1) = \frac{a}{\cosh(b\,x_1 + c)}$ is a global positive C^∞ solution of the ODE for every $a > 0$, b and c.
- (iii) $(m = 3)$ There exist a positive C^∞ solution f, and a positive periodic solution f on $\mathbb{R}$ of the ODE.
- Thm Let $m = 4$. Then, the identity map

$$\mathrm{id} : (\mathbb{R}^4, \tfrac{a}{\cosh(b\,x_1 + c)}\, g_0) \to (\mathbb{R}^4, g_0),$$

  is a proper biharmonic map. Here, $(x_1, \cdots, x_4)$ is the standard coordinate of $\mathbb{R}^4$.

Theorems

- Thm Let $\varphi : (M^2, g) \to (N^{n-1}, h)$ be any harmonic map $(n \geq 2)$.
  For a positive periodic solution f of

$$f^2 f''' - 8\, f f' f'' + 9 f'^3 = 0,$$

  let $f(x, t) := f(t)$, $(x, t) \in M \times S^1$, and $\widetilde{\varphi} : M \times S^1 \ni (x, t) \mapsto (\varphi(x), t) \in N \times S^1$. Then,
- $\widetilde{\varphi} : (M \times S^1, f^2 (g + dt^2)) \to (N \times S^1, h + dt^2)$ is a proper biharmonic map.
- In the case $m = 4$, for $a > 0$, $b, c \in \mathbb{R}$, $\widetilde{\varphi} : (M \times \mathbb{R}, \frac{a}{\cosh(bt + c)} (g + dt^2)) \to (N \times \mathbb{R}, h + dt^2)$ is a proper biharmonic map.

- Thm Let $\varphi : (M^2, g)$ be any Riemannian surface. For a positive periodic solution of

$$f^2 f''' - 8 f f' f'' + 9 f'^3 = 0,$$

  let $f(x, t) := f(t)$, $(x, t) \in M \times S^1$. Then,
- (1) the identity map $\mathrm{id} : (M \times S^1, f^2 (g + dt^2)) \to (M \times S^1, g + dt^2)$ is a proper biharmonic map.
- (2) Let $m = 4$. For $a > 0$, $b, c \in \mathbb{R}$, the identity map $\mathrm{id} : (M \times \mathbb{R}, \frac{a}{\cosh(bt + c)} (g + dt^2)) \to (M \times \mathbb{R}, g + dt^2)$ is a proper biharmonic map.

Biharmonic maps into compact Lie groups

- Let us recall the theories of harmonic maps into Lie groups:
  (1) K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom., **30** (1989), 1–50.
  (2) J. C. Wood, *Harmonic maps into symmetric spaces and integrable systems*, In: *Harmonic Maps and Integrable Systems* eds. by A. P. Fordy and J. C. Wood, Vieweg, 1993, 29–55.
- We want to extend them to biharmonic maps into compact Lie groups

Biharmonic map equations (1)

- Let G be a compact Lie group, and h a bi-invariant Riemannian metric on G corresponding to Ad(G)-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$.
- Let θ be the Maurer-Cartan form on G which is defined by $\theta_y(Z_y) = Z$ $(Z \in \mathfrak{g},\ y \in G)$.
- For a C^∞ map $\psi : M \to G$, let $\alpha := \psi^*\theta$.
- Then, the tension field $\tau(\psi) \in \Gamma(\psi^{-1}TG)$ is given by

$$\langle \theta, \tau(\psi)\rangle = \theta \circ \tau(\psi) = -\delta\alpha,$$

i.e., $\theta_{\psi(x)}(\tau(\psi)(x)) = -(\delta\alpha)_x \quad (x \in G).$

Biharmonic map equations (2)

- Calculate the bitension field :

$$\theta(\tau_2(\psi)) = \theta(J_\psi(\tau(\psi))).$$

- Thm For a C^∞ map $\psi : (M, g) \to (G, h)$,

$$\theta(J_\psi(\tau(\psi))) = -\delta_g\, d\,(\delta\alpha) - \mathrm{Trace}_g([\alpha,\ d\,\delta\alpha]).$$

- Cor (1) $\psi : (M, g) \to (G, h)$ is harmonic $\iff \delta\alpha = 0$.
  (2) $\psi : (M, g) \to (G, h)$ is biharmonic $\iff \delta_g\, d\,\delta\alpha + \mathrm{Trace}_g([\alpha, d\delta\alpha]) = 0.$

Biharmonic maps into Lie groups and integrable systems

- In the following, we consider a C^∞ map

$$\psi : (\mathbb{R}^2, g) \supset \Omega \to (G, h),$$

where $g := \mu^2 g_0$ with $\mu > 0$, a C^∞ function on Ω, G, a compact linear Lie group, and h, a bi-invariant Riemannian metric corresp. to the Ad(G)-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$.
- Then, we have

$$\alpha := \psi^*\theta = \psi^{-1}d\psi.$$

Harmonic map equations

- If we put $A_x := \psi^{-1}\frac{\partial\psi}{\partial x}$, $A_y := \psi^{-1}\frac{\partial\psi}{\partial y}$, we have $\delta\alpha = -\mu^{-2}\left\{\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right\}$.
- Then, ψ is harmonic if and only if $\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y = 0$.
- A_x and A_y are $\mathfrak{g}$-valued 1-forms on Ω, and satisfy the integrability condition:

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x, A_y] = 0.$$

Conversely, if A_x and A_y satisfy the above, then there exists a harmonic map $\psi : \Omega \to (G, h)$ with $\psi^{-1}\frac{\partial\psi}{\partial x} = A_x$ and $\psi^{-1}\frac{\partial\psi}{\partial y} = A_y$.

Biharmonic map equations

- Thm (1) ψ is biharmonic is and only if

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\delta\alpha) - \frac{\partial}{\partial x}[A_x, \delta\alpha] - \frac{\partial}{\partial y}[A_y, \delta\alpha] = 0.$$

- (2) If we define the $\mathfrak{g}$-valued 1-form β by

$$\beta := [A_x, \delta\alpha]\,dx + [A_y, \delta\alpha]\,dy,$$

then, $\delta\beta = -\mu^{-2}\left(\frac{\partial}{\partial x}[A_x, \delta\alpha] + \frac{\partial}{\partial y}[A_y, \delta\alpha]\right)$.
- (3) Thus, ψ is biharmonic if and only if

$$\delta(d\delta\alpha - \beta) = 0.$$

Complexifications

- Take the complex coordinate $z = x + iy$ $(i = \sqrt{-1})$. Then, $dz = dx + idy$, $d\bar{z} = dx - idy$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
- Extend α to a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form on Ω as

$$\alpha = A_x\,dx + A_y\,dy = A_z\,dz + A_{\bar{z}}\,d\bar{z}.$$

Then,

$$-\delta\alpha = \mu^{-2}\left(\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right) = 2\mu^{-2}\left(\frac{\partial}{\partial \bar{z}}A_z + \frac{\partial}{\partial z}A_{\bar{z}}\right),$$

the integrability : $\frac{\partial}{\partial z}A_{\bar{z}} - \frac{\partial}{\partial \bar{z}}A_z + [A_z, A_{\bar{z}}] = 0.$

Harmonic and biharmonic conditions

- Let $\psi : (\mathbb{R}^2, g) \supset \Omega \to (G, h)$ with $g = \mu^2 g_0$.
- ψ is harmonic if and only if $\frac{\partial}{\partial \bar{z}} A_z + \frac{\partial}{\partial z} A_{\bar{z}} = 0$.
- ψ is biharmonic if and only if $\frac{\partial}{\partial \bar{z}} B_z + \frac{\partial}{\partial z} B_{\bar{z}} = 0$.
- Here, $B = B_z\, dz + B_{\bar{z}}\, d\bar{z}$ is a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form on Ω defined by

$$\begin{cases} B_z := \dfrac{\partial}{\partial z}(\delta\alpha) - [A_z, \delta\alpha], \\ B_{\bar{z}} := \dfrac{\partial}{\partial \bar{z}}(\delta\alpha) - [A_{\bar{z}}, \delta\alpha], \end{cases}$$

where $\delta\alpha = -2\mu^{-2}\left(\frac{\partial}{\partial \bar{z}} A_z + \frac{\partial}{\partial z} A_{\bar{z}}\right)$.

Solving biharmonic map equation (1)

- Step 1: Solve the harmonic map equation:

$$(1)\ \frac{\partial}{\partial \bar{z}} B_z + \frac{\partial}{\partial z} B_{\bar{z}} = 0,\ \frac{\partial}{\partial z} B_{\bar{z}} - \frac{\partial}{\partial \bar{z}} B_z + [B_z, B_{\bar{z}}] = 0.$$

- Step 2: For such B, solve A of the P.D.E's (2):

$$\begin{cases} \dfrac{\partial}{\partial z}(\delta\alpha) - [A_z, \delta\alpha] = B_z,\ \dfrac{\partial}{\partial \bar{z}}(\delta\alpha) - [A_{\bar{z}}, \delta\alpha] = B_{\bar{z}}, \\ \dfrac{\partial}{\partial z} A_{\bar{z}} - \dfrac{\partial}{\partial \bar{z}} A_z + [A_z, A_{\bar{z}}] = 0, \end{cases}$$

where $\delta\alpha := -2\mu^{-2}\left(\frac{\partial}{\partial \bar{z}} A_z + \frac{\partial}{\partial z} A_{\bar{z}}\right)$.

Solving biharmonic map equation (2)

- Step 3: For such $A = A_z dz + A_{\bar{z}} d\bar{z}$, solve a C^∞ mapping $\psi : \Omega \to G$ satisfying that

$$\begin{cases} \psi(x_0, y_0) = a \in G, \\ \psi^{-1}\dfrac{\partial \psi}{\partial z} = A_z,\ \psi^{-1}\dfrac{\partial \psi}{\partial \bar{z}} = A_{\bar{z}}. \end{cases}$$

- Then, we have:

Thm This map $\psi : (\Omega, g) \to (G, h)$ is biharmonic. Every biharmonic map can be obtained in this way. ($g := \mu^{-2} g_0$ and μ is a positive C^∞ function on Ω).

Biharmonic map: $\psi : (S^2, g_0) \to (G, h)$

- Thm (Sacks & Uhlenbeck) Every harmonic map $\psi : (\mathbb{R}^2, g) \to (G, h)$ with finite energy can be uniquely extended to a harmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$.
  Conversely, every harmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$ can be obtained in this way.
- "Thm"
  Every biharmonic map $\psi : (\mathbb{R}^2, g) \to (G, h)$ with finite bienergy can be uniquely extended to a biharmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$.
  Conversely, every biharmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$ can be obtained in this way.

Bubbling of biharmonic maps (with N. Nakauchi)

- Thm Let (M, g), (N, h) be compact Riem. mfds. For any $C > 0$, let
  $\mathcal{F} = \{\varphi : (M^m, g) \to (N^n, h)$ biharmonic | $\int_M |d\varphi|^m v_g \leq C$ & $\int_M |\tau(\varphi)|^2 v_g \leq C\}$.
- Then, $\forall \{\varphi_i\} \in \mathcal{F}$, $\exists S = \{x_1, \cdots, x_\ell\} \subset M$, and $\exists$ a biharmonic map $\varphi_\infty : (M \backslash S, g) \to (N, h)$ s.th.
  (1) $\varphi_{i_j} \to \varphi_\infty$ in the C^∞-topology on $M \backslash S$ ($j \to \infty$),
  (2) Radon meas. $|d\varphi_{i_j}|^m v_g$ converges to a meas.

$$|d\varphi_\infty|^m v_g + \sum_{i=1}^{\ell} a_k \delta_{x_k} \quad (j \to \infty).$$

Biharmonic maps into symmetric spaces

- Now let us recall the famous work of Dorfmeister, Pedit and Wu:
  Weierstrass type representation of harmonic maps into symmetric spaces, Commun. Anal. Geom., Vol. 6, No. 4 (1998), 633–668
- which gave a systematic scheme for constructing all harmonic maps from a Riemann surface Σ into G/K.
- We want to extend it to biharmonic maps.

Framework of biharmonic maps into symmetric spaces (1)

- Let (M, g) be a compact Riemannian manifold, $(N, h) = (G/K, h)$, a Riemannian symmetric space with G-invariant Riemannian metric h on G/K, and $\pi : G \to G/K$, the natural projection.
- Let $\varphi : M \to G/K$, a C^∞ map with a local lift $\psi : M \to G$, i.e., $\varphi = \pi \circ \psi$.
- Let θ be the Maurer-Cartan form on G, i.e., $\theta_y(Z_y) = Z$, $Z \in \mathfrak{g}$, $y \in G$.

Framework of biharmonic maps into symmetric space (2)

- Let us consider a $\mathfrak{g}$-valued 1-form α on M given by $\alpha := \psi^*\theta$, and, corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, decompose it as

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}.$$

- Then, the tension field $\tau(\varphi)$ is given by

$$\iota_{\psi(x)^{-1}*}\tau(\varphi) = -\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)],$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M, g) ($\dim M = m$), and δ is the co-differentiation.

Framework of biharmonic maps into symmetric spaces (3)

- Thm The bi-tension field $\tau_2(\varphi)$ of $\varphi : (M, g) \to (G/K, h)$ is given by

$$\tau_2(\varphi) = \Delta_g\left(-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\right)$$
$$+ \sum_{s=1}^{m}\left[\left[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\right], \alpha_{\mathfrak{m}}(e_s)\right].$$

Framework of biharmonic maps into symmetric spaces (4)

- Cor Let $(G/K, h)$ be a Riemannian symmetric space, $\varphi : (M, g) \to (G/K, h)$, a C^∞ map. Then,
- (1) φ is harmonic iff $-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^m[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0$.
- (2) φ is biharmonic iff the following equation holds (#): $\Delta_g\left(-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^m[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\right)$

$$+ \sum_{s=1}^{m}\left[\left[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\right], \alpha_{\mathfrak{m}}(e_s)\right]$$
$$= 0.$$

The above are based on our recent works:

- H. Naito and H. Urakawa, Conformal geometry and biharmonic maps, in preparation, 2009.
- H. Urakawa, Biharmonic maps into compact Lie groups and the integrable systems, 2009, arXiv: 0910.0692.
- N. Nakauchi and H. Urakawa, *Removable singularities and bubling of harmonic maps and biharmonic maps*, preprint, arXiv: 0912.4086.
- H. Urakawa, Biharmonic maps into symmetric spaces and the integrable systems, a preprint, 2011.

Thank you very much for your attention!