SAINT-VENANT AND NAVIER-STOKES EQUATIONS

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ABSTRACT. The two-constants theory is the one now accepted for isotropic, linear elasticity. The original Navier-Stokes equations [NS equations] or Navier equations were introduced in deducing the two-constants theory. From the view of NS equations, we would like to report the deduction of tensor or equations by Navier (§3.1), Poisson (§3.2) , Cauchy (§3.3), Saint-Venant (§3.4), Stokes (§3.5) and the concurrence between each other (§4). Especially, we would like to take up a subject for discussion on Saint-Venant, who is not familliar to the mathematical academy in Japan, however his idea on tensor is, we think, an epock-making for taking the concurrence among three pioneers of NS equations and contributing to Stokes' tensor and equations, which strengthens the frame of NS equations.

CONTENTS

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References

1. Back ground : the Navier-Stokes equations as the productions introduced in the prime of 2nd period of molecules

We can name the period of 1687-1750 : between Newton[31] and Laplace[21] as "*the first period of molecules*". (This classification is due to C.Truesdell, whose editor's introduction to *Leonhardi Euleri Opera Omnia*).[1] Before Laplace, Newton's physics dominated in also fluid mechanics, whose "*Philosophiae naturalis principia mathematica*"[31] was published in 1687. The original Navier-Stokes equations or Navier equations are one of the productions introduced in the prime of this period.

The heated dispute among Poisson, Navier, Arago, Cournot over the then current topics on actions of molecules were published in several journals, above all, on *Annales de chimie et de physique* (ACP) in 1828-29, which was started by Navier[28] against Poisson[43] and finished by Arago[2], (Above all, Poisson[46, 47] and Navier[25, 27] are the main papers or monographies on fluid.) A.Cournot[9] also critisized Navier, and Navier argued against Poisson and Arago in another journal (BSM). Cauchy [6], Saint-Venant[52] and Stokes [55] also proposed their tensors and equations in 1828, 1843 and 1845 respectively.

We show giving and recieving of tensors in the prototypical Navier-Stokes equations in (fig.1).

(fig.1) *Giving and recieving of tensors in the prototypical Navier-Stokes equations*

Navier[26, 27] $(5\text{-}4)_{Ne}$, $(6\text{-}1)_{Ne}$, (2)

$\nearrow$ $\|$ $\Updownarrow$ $\searrow$

Euler $\Longrightarrow$ $\cdots$ $\|$ Poisson[45, 47] $(7\text{-}7)_{Pf}$ $\Longrightarrow$ Saint-Venant[52] (94) $\Longrightarrow$ Stokes[55] (101)

$\searrow$ $\Downarrow$ $\Downarrow$ $\nearrow$

Cauchy[5, 6] $(60)_C$

$\leftarrow$ molecular deduction$\rightarrow$ $\|$ $\leftarrow$ non-molecular deduction$\rightarrow$

2. Two parameters in elastic solid and fluid equations by Navier, Poisson, Cauchy Saint-Venant and Stokes

In our contents, we consider the following main papers and introduce the tables from Table 1 to Table 6, in which we show the equations, tensors and relations of coefficients in each other. We show the equation-numbers of the original papers in the left-hand side of our paper such as :

- $(*)_{Pe}$: Poisson [45],
- $(*)_{Pf}$: Poisson [47], in which contains elastic and fluid,
- $(*)_{Ne}$: Navier [26],
- $(*)_{Nf}$: Navier [27],
- $(*)_C$: Cauchy [6],
- $(*)_{SV}$: Saint-Venant [52],
- $(*)_S$: Stokes [55].

In our paper, the right hand-side equation-numbers are according to our numbering. The following marks show in our paper :

- § $*$: the article number in the original,
- ¶ $*$: the paragraph number in our counting.

[2] Now, we can summarize such as :

- The partial differential equations of the elastic solid or elastic fluid are expressed by using one or the pair of C_1 and C_2 such that : in the elastic solid :

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}.$$

[1] [58], Ser.2 Vol.12, p.81.

[2] In our paper, we cite the description of t_{ij} of the tensor : of Poisson and Cauchy, from C.Truesdell[57], of Navier, from G.Darrigol [12]. in else case by ourself or Schlichting[54].

In the elastic fluid :

$$\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f},$$

where $T_1,\ T_2, \cdots$ are the tensors or terms consisting our equations. For example, in modern notation of the incompressible Navier-Stokes equations, the kinetic equation and the equation of continuity are conventionally described as follows :

$$\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{1}$$

- Moreover, C_1 and C_2 are described as follows :

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 L r_1 g_1 = \frac{2\pi}{15}\mathcal{L} r_1 g_1, \\ C_2 = C_4 L r_2 g_2 = \frac{2\pi}{3} L r_2 g_2. \end{cases}$$

- C_1 and C_2 are two coefficients, for example, k and K by Poisson, or ε and E by Navier, or R and G by Cauchy, and which are expressed by the infinite operator $\mathcal{L}$ ($\sum_0^\infty$ or $\int_0^\infty$) by personal principles or methods, where r_1 and r_2 are the functions related to the radius of the active sphere of the molecules, rised to the power of n, for Poisson's and Navier's case, the relation of function in expressing by logarithm to the base of r exists such that : $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are the certain functions which are dependent on r and are described with attraction &/or repulsion.
- S_1 and S_2 are the two expressions which describe the surface of active unit-sphere at the center of a molecule by the double integral (or single sum in case of Poisson's fluid).
- g_3 and g_4 are certain compound trianglar-functions to compute the moment in the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients from the invariant tensor. Except for Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are computed from the total moment of the active sphere of the molecules in computing only by integral, and which are independent on personal manner. In Poisson's case, after multiplying by $\frac{1}{4\pi}$, we get the same as above.
- The ratio of the two coefficients including Poisson's case is always same as : $\frac{C_3}{C_4} = \frac{1}{5}$.

3. Principles and deduction to equations or tensor

3.1. Naviers' principle and equations.

3.1.1. From Euler to Navier.

Navier-Stokes equations on the incompressible fluid(1). His equations are as follw :

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases} \tag{2}$$

and the equation of continuity :

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{3}$$

He explains ε from various concepts in [25, p.251] :

> ε is the constant which we mentioned above. Many experiments teach that this constant takes the various values for each fluids, and varies with the temperature for each fluid. It is considerd also as variant with the pression ; but we have observed as the known facts, on the contrary, that ε is almostly independ of the force which tends to compress the partial differences of the fluid.

Navier cites the Euler's equations ([26, p.399]) :

$$\begin{cases} P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right), \\ Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right), \\ R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right), \end{cases}$$

TABLE 1. C_1, C_2, C_3, C_4 : the constant of definitions and computing of total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name	elastic solid	elastic fluid	polar system
1	Poisson [45, 47]	$C_1 = k \equiv \frac{2\pi}{15}\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3}\sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta\ g_3 \Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta\ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot,\cdot\}$	$C_1 = -k \equiv -\frac{1}{30\epsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15}\sum \frac{r^3}{4\pi\epsilon^3}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\epsilon^3}\sum rfr$ $= -\frac{2\pi}{3}\sum \frac{r}{4\pi\epsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3\text{-}2)_{Pf}\quad N = \frac{1}{6\epsilon^3}\sum rfr \Rightarrow \frac{1}{6}$	$x_1 = r\cos\beta\cos\gamma.$ $y_1 = r\sin\beta\sin\gamma,$ $\zeta = -r\cos\beta$
2	Navier [26, 27]	$C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi\ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = \frac{2\pi}{15}$	$C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3}\int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_4 \Rightarrow \frac{2\pi}{3}$	$\alpha = \rho\cos\psi\cos\varphi,$ $\beta = \rho\cos\psi\sin\varphi,$ $\gamma = \rho\sin\psi$
3	Cauchy [6]	$C_1 = R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr$ $= \pm\frac{2\pi\Delta}{15}\int_0^\infty [r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)]dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3 \mathrm{f}(r)dr$ $C_3 = \frac{1}{2}\int_0^{2\pi} \cos^2 qdq \int_0^\pi \cos^2\alpha\cos^2\beta dp,$ $= \frac{1}{2}\int_0^{2\pi}\cos^2 qdq \int_0^\pi \cos^2 p \sin^2 p \sin pdp = \frac{2\pi}{15},$ $C_4 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi \cos^2\alpha \sin pdqdp$ $= \pi\int_0^\pi \cos^2 p\sin\ pdp = \frac{2\pi}{3},$	samely as in elastic solid	$\cos\alpha = \cos p,$ $\cos\beta = \sin p\cos q,$ $\cos\gamma = \sin p\sin q$
4	Saint-Venant[52]		$C_1 = \varepsilon,\quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes[55]	$C_1 = A,\quad C_2 = B$	$C_1 = \mu,\quad C_2 = \frac{\mu}{3}$	

TABLE 2. The expression of the total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name	problem	C_1	C_2	C_3	C_4	L	r_1	r_2	g_1	g_2	S_1, S_2, g_3, g_4	remark
1	Poisson [45]	elastic solid	k		$\frac{2\pi}{15}$		$\sum\frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		cf.Table 3	
				K		$\frac{2\pi}{3}$	$\sum\frac{1}{\alpha^5}$		r^3		fr		
2	Poisson [47]	fluid	k		$\frac{1}{30}$		$\sum\frac{1}{\epsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$			$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum\frac{1}{\epsilon^3}$		r		fr		$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
3	Navier [26]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$			ρ : radius
4	Navier fluid [27]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$			ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$		
5	Cauchy [6]	system of particles	R		$\frac{2\pi}{15}$		$\int_0^\infty dr$	r^3		$f(r)$			$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}$	$\int_0^\infty dr$		r^3		$\mathrm{f}(r)$		$f(r) \neq \mathrm{f}(r)$
6	Saint-Venant [52]	fluid	ε	$\frac{\varepsilon}{3}$									
7	Stokes [55]	fluid	μ	$\frac{\mu}{3}$									
8	Stokes [55]	elastic solid	A	B									A=5B

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{4}$$

3.1.2. **Principles and means of constant ε in elastic solid.**
From Navier[26, p.386], we cite his context about the computing of moments of total forces by integral

TABLE 3. S_1, S_2, g_3, g_4 : the triangular functions computing of total moment of molecular actions in unit sphere by Poisson, Navier, Cauchy & Stokes

<table>
<tr><th>no</th><th>name</th><th>S_1, S_2, g_3, g_4</th></tr>
<tr><td>1</td><td>Poisson</td><td>

$\cdot g_3$ and g_4 are in the following tensor :

$$\begin{cases} g = a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta, & g' = g\frac{du}{dx} + h\frac{du}{dy} + l\frac{du}{dz}, \\ h = a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta, & h' = g\frac{dv}{dx} + h\frac{dv}{dy} + l\frac{dv}{dz}, \\ l = a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta, & l' = g\frac{dw}{dx} + h\frac{dw}{dy} + l\frac{dw}{dz} \end{cases}$$

$$\begin{cases} P = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(g+g')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')g\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(h+h')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')h\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(l+l')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')l\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases}$$

i.e.

$$\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\Delta\left(\begin{bmatrix} g+g' & (gg'+hh'+ll')g \\ h+h' & (gg'+hh'+ll')h \\ l+l' & (gg'+hh'+ll')l \end{bmatrix}\begin{bmatrix} \sum\frac{r^3 fr}{\alpha^5} \\ \sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix}\right)$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} d\beta\, d\gamma\left(\begin{bmatrix} g_4 & g_3 \end{bmatrix}\begin{bmatrix} K' \\ k' \end{bmatrix}\right),$$

where $\Delta := \cos\beta\cdot\sin\beta\ d\beta\ d\gamma$, $K' := \sum\frac{r^3 fr}{\alpha^5}$, $k' := \sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$.

$\cdot S_1$ and S_2 are given from above.

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<tr><td>2</td><td>Navier elastic solid</td><td>

$\cdot g_3$:

$g_3 = \frac{1}{2}\delta f^2$

$f \equiv \rho\Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi + \left(\frac{dx}{db}+\frac{dy}{da}\right)\cos^2\psi\sin\varphi\cos\varphi + \left(\frac{dx}{dc}+\frac{dz}{da}\right)\cos\psi\sin\psi\cos\varphi$
$+\frac{dy}{db}\cos^2\psi\sin^2\varphi + \left(\frac{dy}{dc}+\frac{dz}{db}\right)\sin\psi\cos\psi\sin\varphi + \frac{dz}{dc}\sin^2\psi\Big].$

</td></tr>
<tr><td>3</td><td>Navier fluid</td><td>

$\cdot g_3$:

$\alpha = \rho\cos\psi\cos\varphi,\quad \beta = \rho\cos\psi\sin\varphi,\quad \gamma = \rho\sin\psi$

$$g_3 = V\delta V = \left[\alpha\left(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\right) + \beta\left(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\right) + \gamma\left(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\right)\right] \times \left[\alpha\left(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\right) + \beta\left(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\right) + \gamma\left(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\right)\right],$$

$\cdot g_4$:

$$g_4 = V\delta\ V = \begin{cases} \alpha'^2 & \{ (u\sin^2 r - v\sin r\cos r)\delta u, \\ & (-u\sin r\cos r + v\cos^2 r)\delta v \}, \\ \beta'^2 & \{ (u\cos^2 r\sin^2 s + v\sin r\cos r\sin^2 s + w\cos r\sin s\cos s)\delta u, \\ & (u\sin r\cos r\sin^2 s + v\sin^2 r\sin^2 s + w\sin r\sin s\cos s)\delta v, \\ & (u\cos r\sin s\cos s + v\sin r\sin s\cos s + w\cos^2 s)\delta w \}, \\ \gamma'^2 & \{ (u\cos^2 r\cos^2 s + v\sin r\cos r\cos^2 s - w\cos r\sin s\cos s)\delta u, \\ & (u\sin r\cos r\cos^2 s + v\sin^2 r\cos^2 s - w\sin r\sin s\cos s)\delta v, \\ & (-u\cos r\sin s\cos s - v\sin r\sin s\cos s + w\sin^2 s)\delta w \} \end{cases}$$

where $\alpha' = \rho\cos\psi\cos\varphi,\quad \beta' = \rho\cos\psi\sin\varphi,\quad \gamma' = \rho\sin\psi.$

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<tr><td>3</td><td>Cauchy</td><td>

$\cdot g_3 = g_4 = \frac{v}{2}$:

$$(44)_C\ \begin{cases} G = G\left(\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1\right) \equiv\ GA_1, \\ L = L\left(\cos^4\alpha_1 + \cos^4\beta_1 + \cos^4\gamma_1\right) \\ \quad +6R\left(\cos^2\beta_1\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_1\right) \equiv LB + 6RC, \\ R = R\left(\cos^2\beta_1\cos^2\gamma_2 + \cos^2\beta_2\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_2\right. \\ \quad \left.+\cos^2\gamma_2\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1\right) \\ \quad +4R\left(\cos\beta_1\cos\beta_1\cos\gamma_1\cos\gamma_2 + \cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2\right. \\ \quad \left.+\cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2\right) \\ \quad +L\left(\cos^2\alpha_1\cos^2\alpha_2 + \cos^2\beta_1\cos^2\beta_2 + \cos^2\gamma_1\cos^2\gamma_2\right) \equiv RD + 4RE + LF \end{cases}$$

where $\begin{cases} \cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1 = 1, \quad \cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2 = 1, \\ \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0 \end{cases}$

$\cdot$ From $(49)_C$ $G = \frac{\Delta}{2}\mathbf{S}[\pm r\cos^2\alpha f(r)v]$, $R = \frac{\Delta}{2}\mathbf{S}[r\cos^2\alpha\cos^2\beta f(r)v]$

and $(50)_C$ $\begin{cases} G = \pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\sin p\,dr\,dq\,dp, \\ R = \frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\cos^2\beta\sin p\,dr\,dq\,dp \end{cases}$

$$\begin{cases} S_1 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\cos^2\alpha\cos^2\beta\sin\ p\,dp = \frac{1}{2}\int_0^{2\pi}\cos^2 q\,dq\int_0^\pi \cos^2 p(1-\cos^2 p)\sin p\,dp \\ \quad = \frac{1}{2}\pi\left(\frac{2}{3}-\frac{2}{5}\right) = \frac{2\pi}{15} \equiv C_3, \\ S_2 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\cos^2\alpha\sin p\,dq\,dp = \frac{1}{2}2\pi\int_0^\pi\cos^2 p\sin\ p\,dp = \frac{2\pi}{3} \equiv C_4. \end{cases}$$

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TABLE 4. $T_1\&T_2$: tensors & equations by Navier, Poisson, Cauchy & Stokes in elastic solid

no	name	tensor & equations
1	Navier elastic solid [26]1821-27	· Navier's tensor : $t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ · Navier's equations : $\begin{cases} X - \frac{d^2u}{dt^2} + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dydx} + 2\frac{d^2w}{dzdx}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + \varepsilon\left(\frac{d^2u}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dzdy}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2w}{dxdz} + 2\frac{d^2w}{dydz}\right) = 0 \end{cases}$ Same as $(6)_{Pe}$ which is deduced from Poisson when $K = 0$ and $a^2 = \frac{3k}{\rho}$. · Indeterminate equations with ε : $(5\text{-}1)_{Ne}$ $0 = \varepsilon \iiint da\, db\, dc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dx}{da} \\ + \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dx}{da}\frac{\delta dz}{dc} + \frac{dz}{dc}\frac{\delta dx}{da} + 3\frac{dy}{db}\frac{\delta dy}{db} \\ + \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} \end{cases}$ $- \iiint da\, db\, dc\, (X\delta x + Y\delta y + Z\delta z) - \int ds\, (X'\delta x' + Y'\delta y' + Z'\delta z')$. · Boundary condition with ε : $(5-4)_{Ne}$ $\begin{cases} X' = \varepsilon\left[\cos l\left(3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'}\right) + \cos m\left(\frac{dx'}{db'} + \frac{dy'}{da'}\right) + \cos n\left(\frac{dx'}{dc'} + \frac{dz'}{da'}\right)\right], \\ Y' = \varepsilon\left[\cos l\left(\frac{dx'}{db'} + \frac{dy'}{da'}\right) + \cos m\left(\frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'}\right) + \cos n\left(\frac{dy'}{dc'} + \frac{dz'}{db'}\right)\right], \\ Z' = \varepsilon\left[\cos l\left(\frac{dx'}{dc'} + \frac{dz'}{da'}\right) + \cos m\left(\frac{dy'}{dc'} + \frac{dz'}{db'}\right) + \cos n\left(\frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'}\right)\right]. \end{cases}$
2	Poisson elastic solid [45]1828-29	· Poisson's tensor : $t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ · Poisson's equations : $\begin{cases} P = -K\left(c + \frac{du}{dx}c + \frac{du}{dy}c' + \frac{du}{dz}c''\right) - k\left(3\frac{du}{dx}c + \frac{du}{dy}c' + \frac{du}{dz}c'' + \frac{dv}{dx}c' + \frac{dv}{dy}c + \frac{dw}{dx}c'' + \frac{dw}{dz}c\right), \\ Q = -K\left(c' + \frac{dv}{dx}c + \frac{dv}{dy}c' + \frac{dv}{dz}c''\right) - k\left(\frac{dv}{dx}c + 3\frac{dv}{dy}c' + \frac{dv}{dz}c'' + \frac{du}{dx}c' + \frac{du}{dy}c + \frac{dw}{dy}c'' + \frac{dw}{dz}c'\right), \\ R = -K\left(c'' + \frac{dw}{dx}c + \frac{dw}{dy}c' + \frac{dw}{dz}c''\right) - k\left(\frac{dw}{dx}c + \frac{dw}{dy}c' + 3\frac{dw}{dz}c'' + \frac{du}{dx}c'' + \frac{du}{dz}c + \frac{dv}{dy}c'' + \frac{dv}{dz}c'\right). \end{cases}$ If $K = 0 \Rightarrow$ $(6)_{Pe}$ $\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2w}{dxdz} + \frac{2}{3}\frac{d^2w}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = 0, \end{cases}$ where $a^2 = \frac{3k}{\rho}$, these are equal to Navier's elastic solid equations.
3	Cauchy [6]1828	· Cauchy's tensor : $t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ · Cauchy's equations : $(68)_C$ $\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$ where $L = M = N$, $P = Q = R$. If $G(= H = I) = 0 \Rightarrow$ $(84)_C$ $\begin{cases} L\frac{\partial^2\xi}{\partial x^2} + R\frac{\partial^2\xi}{\partial y^2} + Q\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ R\frac{\partial^2\eta}{\partial x^2} + M\frac{\partial^2\eta}{\partial y^2} + P\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ Q\frac{\partial^2\zeta}{\partial x^2} + P\frac{\partial^2\zeta}{\partial y^2} + N\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$ At last, it rests only G and R.

• ¶ 4.

$$(3\text{-}5)_{Ne} \quad \sqrt{\alpha^2+\beta^2+\gamma^2} + \frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\left[\frac{dx}{da}\alpha^2 + \left(\frac{dx}{db}+\frac{dy}{da}\right)\alpha\beta + \left(\frac{dx}{dc}+\frac{dz}{da}\right)\alpha\gamma + \frac{dy}{db}\beta^2 + \left(\frac{dy}{dc}+\frac{dz}{db}\right)\beta\gamma + \frac{dz}{dc}\gamma^2\right]$$

Le premier terme est la valeur primitive de la distance MM' des deux points que l'on considère, qui a été représentée ci-dessuus pa ρ. Le second terme représente donc la variation que cette distance a subie par suite du changement de figure du corps, et à laquelle la force qui agit de M' sur M est propotionnelle. Si on remplace α, β, γ par le valeurs

$$\begin{cases} \alpha = \rho\cos\psi\cos\varphi, \\ \beta = \rho\cos\psi\sin\varphi, \\ \gamma = \rho\sin\psi, \end{cases}$$

TABLE 5. The tensors & equations by Navier, Poisson, Saint-Venant & Stokes in fluid

no	name	tensor & equations
1	Navier incomp. fluid [27]1822-27	· Navier's tensor : $t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ · Navier's equations : $\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\Big(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\Big(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w ; \end{cases}$ i.e. $\frac{\partial \mathbf{u}}{\partial t} - \varepsilon\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}$, div $\mathbf{u} = 0$ · Indeterminate equations with ε and E : $0 = \iiint dxdydz \begin{cases} \big[P - \frac{dp}{dx} - \rho\Big(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\Big)\big]\delta u \\ \big[Q - \frac{dp}{dy} - \rho\Big(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\Big)\big]\delta v \\ \big[R - \frac{dp}{dz} - \rho\Big(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\Big)\big]\delta w \end{cases}$ $-\varepsilon \iiint dxdydz \begin{cases} 3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz} + \frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy} + \frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz} \\ \frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx} + \frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz} + \frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy} \\ \frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx} + \frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy} + \frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz} \end{cases}$ $+\mathbf{S}ds^2E(u\delta u + v\delta v + w\delta w)$. · Boundary condition with ε and E : $\begin{cases} Eu + \varepsilon[\cos l 2\frac{du}{dx} + \cos m\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) + \cos n\Big(\frac{du}{dz} + \frac{dw}{dx}\Big)] = 0, \\ Ev + \varepsilon[\cos l\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) + \cos m 2\frac{dv}{dy} + \cos n\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big)] = 0, \\ Ew + \varepsilon[\cos l\Big(\frac{du}{dz} + \frac{dw}{dx}\Big) + \cos m\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) + \cos n 2\frac{dw}{dz}] = 0, \end{cases}$
2	Poisson fluid [47]1829-31	· Poisson's tensor : $t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$: $\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\Big(\frac{du}{dz} + \frac{dw}{dx}\Big) & \beta\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} \\ \beta\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) \\ p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} & \beta\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & \beta\Big(\frac{du}{dz} + \frac{dw}{dx}\Big) \end{bmatrix}$, $(k + K)\alpha = \beta, \quad (k - K)\alpha = \beta', \quad p = \psi t = K, \quad$ then $\quad \beta + \beta' = 2k\alpha$. $\varpi \equiv -\alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}$. · Poisson's equations in incompressible fluid : If ψt(= density), χt(=pressure) $\equiv$ const. and incompressible $\Rightarrow (9)_{Pf} \quad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2}) = 0, \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0. \end{cases}$ i.e. $\frac{\partial \mathbf{u}}{\partial t} + \frac{\beta}{\rho}\Delta\mathbf{u} + \frac{1}{\rho}\nabla\varpi = \mathbf{f}$ · Coincidence with Stokes' equations : by $\varpi = p + \frac{\alpha}{3}(K + k)\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)$ $\Rightarrow \nabla\varpi = \nabla p + \frac{\beta}{3}\nabla\cdot(\nabla\cdot\mathbf{u})$, $(9)_{Pf} \cong (12)_S$ of Stokes.
3	Saint-Venant fluid [52]1834-43	·Saint-Venant's tensor : $t_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ i.e. $\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi + 2\varepsilon\frac{d\xi}{dx}, & \varepsilon\Big(\frac{d\xi}{dy} + \frac{d\eta}{dx}\Big) & \varepsilon\Big(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\Big) \\ \varepsilon\Big(\frac{d\xi}{dy} + \frac{d\eta}{dx}\Big) & \pi + 2\varepsilon\frac{d\eta}{dy} & \varepsilon\Big(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\Big) \\ \varepsilon\Big(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\Big) & \varepsilon\Big(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\Big) & \pi + 2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}$, where $\quad \pi = \frac{1}{3}\Big(P_{xx} + P_{yy} + P_{zz}\Big) - \frac{2\epsilon}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)$.
4	Stokes fluid [55]1845-49	· Stokes' tensor : $-t_{ij} = \{p - 2\mu(v_{i,j} - \delta) + \gamma\}\delta_{ij} - \gamma = (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i})$, where $\quad 3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, $\gamma = \mu(v_{i,j} + v_{j,i})$, $v_{k,k} = \sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}$ c.f.[54, p.58] i.e. $\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p - 2\mu\Big(\frac{du}{dx} - \delta\Big) & -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - 2\mu\Big(\frac{dv}{dy} - \delta\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - 2\mu\Big(\frac{dw}{dz} - \delta\Big) \end{bmatrix}$, where $\quad 3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$. · Stokes' equations with μ : $(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0. \end{cases}$ i.e. $\rho\Big(\frac{D\mathbf{u}}{Dt} - \mathbf{f}\Big) + \nabla p - \mu\Big(\Delta\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\Big) = 0$ i.e. $\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \frac{\mu}{3}\Big(4\frac{d^2u}{dx^2} + 3\frac{d^2u}{dy^2} + 3\frac{d^2u}{dz^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \frac{\mu}{3}\Big(3\frac{d^2v}{dx^2} + 4\frac{d^2v}{dy^2} + 3\frac{d^2v}{dz^2} + \frac{d^2u}{dxdy} + \frac{d^2w}{dydz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \frac{\mu}{3}\Big(3\frac{d^2w}{dx^2} + 3\frac{d^2w}{dy^2} + 4\frac{d^2w}{dz^2} + \frac{d^2u}{dxdz} + \frac{d^2w}{dydz}\Big) = 0, \end{cases}$

TABLE 6. The conditions in reducing to others

no	name	Poisson	Navier	Cauchy	Saint-Venant	cf.
1	Poisson		if $K=0, \varepsilon=\frac{a^3}{3}=\frac{k}{\rho} \Rightarrow \varepsilon \cong k$			$(6)_{Pe}$
2	Navier	even if $K=0 \Rightarrow \varepsilon \neq k$, [29, p.103, § 8]				
3	Cauchy	if $G=0 \Rightarrow R \cong a^2$, [44, p.206, § 1]	if $G=0 \Rightarrow R \cong \varepsilon$, [6, p.251]			$(84)_C$
4	Saint-Venant	if we put $\pi=\varpi-\varepsilon\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)$ then Saint-Venant's ε equals to Navier's, Poisson's and Cauchy's one	same as left	same as left		(94)
5	Stokes	if $\alpha(K+k) \equiv \beta \Rightarrow \mu \cong \beta$ i.e. $\varpi=p+\frac{\alpha}{3}(K+k)\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)$ $\Rightarrow \nabla\varpi=\nabla p+\frac{\beta}{3}\nabla\cdot(\nabla\cdot\mathbf{u}), (9)_{Pf} \cong (12)_S$.	$\neq$, [55, p.77, footnote]			$(12)_S$

cette variation deviendra

$$f \equiv \rho\Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi+\Big(\frac{dx}{db}+\frac{dy}{da}\Big)\cos^2\psi\sin\varphi\cos\varphi+\Big(\frac{dx}{dc}+\frac{dz}{da}\Big)\cos\psi\sin\psi\cos\varphi$$
$$+\ \frac{dy}{db}\cos^2\psi\sin^2\varphi+\Big(\frac{dy}{dc}+\frac{dz}{db}\Big)\sin\psi\cos\psi\sin\varphi+\frac{dz}{dc}\sin^2\psi\Big]. \qquad (5)$$

Représentons pour abréger, cette quqntité par f. La force avec laquelle le point M' attire M sera donc propotionnelle à f. Le moment de cette force, cette expression étant prise dans le même sens que dans la *Mécanique analytique*, sera évidement propotionnel à $f\delta f$, ou à $\frac{1}{2}\delta f^2$. Parconséquent

- si l'on multiplie $\frac{1}{2}\delta f^2$ par $d\rho d\psi d\varphi\rho^2\cos\psi f\rho$;
- si l'on transporte le signe δ en avant des signes d'intégration relatifs à ρ, ψ and φ, ce qui est permis ;
- et si l'on intégre èntre les mêmes limites qu'on l'a fait dans le no 3 :

on aura une quantité propotionnelle à la somme des moments de toutes les forces intérieures par lesquelles le point M est sollicité. Cette quantité est donc

$$(4\text{-}7)_{Ne} \quad \int_0^\infty d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\psi\int_0^{2\pi}d\varphi\rho^4\cos\psi f\rho\Big(\frac{1}{2}\delta f^2\Big)$$
$$= \frac{1}{2}\delta\int_0^\infty d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\psi\int_0^{2\pi}d\varphi\rho^4\cos\psi f\rho$$
$$\times\ \Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi+\Big(\frac{dx}{db}+\frac{dy}{da}\Big)\cos^2\psi\sin\varphi\cos\varphi+\Big(\frac{dx}{dc}+\frac{dz}{da}\Big)\cos\psi\sin\psi\cos\varphi$$
$$+\ \frac{dy}{db}\cos^2\psi\sin^2\varphi+\Big(\frac{dy}{dc}+\frac{dz}{db}\Big)\sin\psi\cos\psi\sin\varphi+\frac{dz}{dc}\sin^2\psi\Big]^2.$$

$$f^2=\Big(\frac{dx}{da}\Big)^2\cos^4\psi\cos^4\varphi+\Big\{\Big(\frac{dx}{db}\Big)^2+\Big(\frac{dy}{da}\Big)^2+2\frac{dx}{db}\frac{dy}{da}\Big\}\cos^4\psi\sin^2\varphi\cos^2\varphi$$
$$+\ \Big\{\Big(\frac{dx}{dc}\Big)^2+\Big(\frac{dz}{da}\Big)^2+2\frac{dx}{dc}\frac{dz}{da}\Big\}\cos^2\psi\sin^2\psi\cos^2\varphi+\Big(\frac{dy}{db}\Big)^2\cos^4\psi\sin^4\varphi$$
$$+\ \Big\{\Big(\frac{dy}{dc}\Big)^2+\Big(\frac{dz}{db}\Big)^2+2\frac{dy}{dc}\frac{dz}{db}\Big\}\sin^2\psi\cos^2\psi\sin^2\varphi+\Big(\frac{dz}{dc}\Big)^2\sin^4\psi$$

Here, we would like to show Navier's mistake. At first we integrate above with respect to φ. By using the formulae with adding to (56) as follows :

$$\begin{cases}\int\cos^n xdx=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}xdx,\\ \int\sin^n xdx=-\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int\sin^{n-2}xdx,\end{cases}$$

then :

$$\int_0^{2\pi} \cos^4 \varphi d\varphi = \int_0^{2\pi} \sin^4 \varphi d\varphi = \frac{3}{4}\pi, \quad \int_0^{2\pi} \sin^2 \varphi \cos^2 \varphi d\varphi = \frac{\pi}{4}, \quad \int_0^{2\pi} \cos^2 \varphi d\varphi = \int_0^{2\pi} \sin^2 \varphi d\varphi = \pi$$

Hence, it follows that : [3]

$$\begin{aligned} &\frac{1}{2}\delta \int_0^{\infty} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} d\varphi \rho^4 \cos\psi f\rho f^2 \\ = \;& \frac{1}{2}\delta \int_0^{\infty} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \\ \times \;& \frac{\pi}{4}\Big[3\frac{d^2x}{da^2}\cos^5\psi + \Big\{\Big(\frac{dx}{db}+\frac{dy}{da}\Big)^2 + 2\frac{dx}{db}\frac{dy}{da}\Big\}\cos^5\psi \\ + \;& 4\Big\{\Big(\frac{dx}{dc}+\frac{dz}{da}\Big)^2 + 2\frac{dx}{dc}\frac{dz}{da}\Big\}\cos^3\psi\sin^2\psi + 3\frac{d^2y}{db^2}\cos^5\psi \\ + \;& 4\Big\{\Big(\frac{dy}{dc}+\frac{dz}{db}\Big)^2 + 2\frac{dy}{dc}\frac{dz}{db}\Big\}\sin^2\psi\cos^3\psi + 8\frac{d^2z}{dc^2}\sin^4\psi\cos\psi\Big]. \end{aligned} \tag{6}$$

Here, we would like to notice our correction of the last term of $[\cdots 8 \cdots]$ in (6) from 3 to 8, however this correction will not give any effect to Navier's description below. Next we integrate above with respect to ψ. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\psi d\psi = \frac{16}{15}, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\psi \sin^2\psi d\psi = \frac{4}{15}, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4\psi\cos\psi d\psi = \frac{2}{5}$$

After representing the coefficient which rest on the front of the integral with respect to ρ with ε, get from (6) :

$$\begin{aligned} \frac{1}{2}\varepsilon\delta\Big[\;& \Big\{3\frac{d^2x}{da^2} + \Big\{\Big(\frac{dx}{db}+\frac{dy}{da}\Big)^2 + 2\frac{dx}{db}\frac{dy}{da}\Big\} + \Big\{\Big(\frac{dx}{dc}+\frac{dz}{da}\Big)^2 + 2\frac{dx}{dc}\frac{dz}{da}\Big\} \\ & +3\frac{d^2y}{db^2} + \Big\{\Big(\frac{dy}{dc}+\frac{dz}{db}\Big)^2 + 2\frac{dy}{dc}\frac{dz}{db}\Big\} + 3\frac{d^2z}{dc^2} \Big] \end{aligned} \tag{7}$$

Here,

$$(3\text{-}9)_{N^e} \quad \varepsilon \equiv \frac{1}{2}\frac{\pi}{4}\frac{16}{15}\int_0^{\infty} d\rho\rho^4 f\rho = \frac{2\pi}{15}\int_0^{\infty} d\rho\rho^4 f\rho \tag{8}$$

This ε of (8) should be multiplied by $\frac{1}{2}$, when the moments of the total forces *in the solid* is computed, namely it becomes the same as (29).

• ¶ 5.

$$\begin{aligned} (5\text{-}1)_{N^e} \quad 0 = \; & \varepsilon \iiint da\, db\, dc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dx}{da} \\ +\frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dx}{da}\frac{\delta dz}{dc} + \frac{dz}{dc}\frac{\delta dx}{da} + 3\frac{dy}{db}\frac{\delta dy}{db} \\ +\frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} \end{cases} \\ - & \iiint da\, db\, dc\, (X\delta x + Y\delta y + Z\delta z) - \int ds\, (X'\delta x' + Y'\delta y' + Z'\delta z'). \end{aligned} \tag{9}$$

When the first term of ε in the right-hand side of (9) is arranged in respect to $\delta x, \delta y$ and δz then :

$$\varepsilon \iiint dadbdc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dy}{db}\frac{\delta dx}{da} + \frac{dz}{da}\frac{\delta dx}{dc} + \frac{dz}{dc}\frac{\delta dx}{da} \\ +\frac{dy}{da}\frac{\delta dy}{da} + 3\frac{dy}{db}\frac{\delta dy}{db} + \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dx}{da}\frac{\delta dy}{db} + \frac{dx}{db}\frac{\delta dy}{da} + \frac{dz}{db}\frac{\delta dy}{dc} + \frac{dz}{dc}\frac{\delta dy}{db} \\ +\frac{dz}{da}\frac{\delta dz}{da} + \frac{dz}{db}\frac{\delta dz}{db} + 3\frac{dz}{dc}\frac{\delta dz}{dc} + \frac{dx}{da}\frac{\delta dz}{dc} + \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dy}{db}\frac{\delta dz}{dc} + \frac{dy}{dc}\frac{\delta dz}{db} \end{cases} \tag{10}$$

[3] Remark : $f\rho$ does not mean $f \times \rho$ but $f(\rho)$. We compute $(\bullet + \bullet)^2$ in (6) as usual, for example : $(\frac{dx}{dc} + \frac{dz}{da})^2 = \frac{d^2x}{dc^2} + 2\frac{dx}{dc}\frac{dz}{da} + \frac{d^2z}{da^2}$.

Moreover, we rearrange (10) for differential : $\frac{\delta x'}{da'}, \frac{\delta x'}{db'}, \frac{\delta x'}{dc'}, \frac{\delta y'}{da'}, \frac{\delta y'}{db'}, \frac{\delta y'}{dc'}, \frac{\delta z'}{da'}, \frac{\delta z'}{db'}, \frac{\delta z'}{dc'}$ as follows :

$$\varepsilon \iiint dadbdc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dy}{db}\frac{\delta dx}{da} + \frac{dz}{dc}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dx}{dc} \\ +\frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + 3\frac{dy}{db}\frac{\delta dy}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dy}{dc} \\ +\frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dx}{da}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} + \frac{dy}{db}\frac{\delta dz}{dc} \end{cases} \tag{11}$$

Using (10) and partial integration of $\delta x, \delta y$ and δz, we make the top term of (12), in which $-$ is leading. And using (11), we show only the first differential order : $\delta x', \delta y', \delta z'$ in the middle term of (12) as follows :

$$\begin{aligned}
&(5\text{-}2)_{Ne} \quad 0 \\
&= -\varepsilon \iiint da\ db\ dc \begin{cases} \left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\right)\delta x \\ +\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\right)\delta y \\ +\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right)\delta z \end{cases} \\
&+ \varepsilon\Big[\iint db'\ dc'\Big(3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'db'\Big(\frac{dx'}{dc} + \frac{dz'}{da'}\Big)\Big]\delta x' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'dc'\Big(\frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'db'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big)\Big]\delta y' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{dc'} + \frac{dz'}{da'}\Big) + \iint da'dc'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big) + \iint da'db'\Big(\frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'}\Big)\Big]\delta z' \\
&- \iiint da\ db\ dc(X\delta x + Y\delta y + Z\delta z) - \int ds(X'\delta x' + Y'\delta y' + Z'\delta z').
\end{aligned} \tag{12}$$

We solve the indeterminate equations (12) [4] of equilibrium in an elastic solid as follows. At first, we get the following two equations from (12) :

- The force inside the solid corps :

$$-\iiint da\ db\ dc(X\delta x + Y\delta y + Z\delta z) = \varepsilon \iiint da\ db\ dc \begin{cases} \left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\right)\delta x \\ +\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\right)\delta y \\ +\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right)\delta z. \end{cases} \tag{13}$$

- The force on the boundary :

$$\begin{aligned}
&\int ds(X'\delta x' + Y'\delta y' + Z'\delta z') \\
&= \varepsilon\Big[\iint db'\ dc'\Big(3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'db'\Big(\frac{dx'}{dc} + \frac{dz'}{da'}\Big)\Big]\delta x' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'dc'\Big(\frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'db'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big)\Big]\delta y' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{dc'} + \frac{dz'}{da'}\Big) + \iint da'dc'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big) + \iint da'db'\Big(\frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'}\Big)\Big]\delta z'.
\end{aligned} \tag{14}$$

$$\Rightarrow \int ds \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \varepsilon \begin{bmatrix} 3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{dc'} + \frac{dz'}{da'} \\ \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dy'}{dc'} + \frac{dz'}{db'}, \\ \frac{dx'}{dc'} + \frac{dz'}{da'} & \frac{dy'}{dc'} + \frac{dz'}{db'} & \frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'} \end{bmatrix} \begin{bmatrix} \iint db'\ dc' \\ \iint da'dc' \\ \iint da'db' \end{bmatrix}, \tag{15}$$

here this tensor is symmetric. From (13), we get the inside forces of the elastic solid as follows :

$$(5\text{-}3)_{Ne} \quad \begin{cases} -X = \varepsilon\Big(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\Big), \\ -Y = \varepsilon\Big(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\Big), \\ -Z = \varepsilon\Big(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\Big), \end{cases} \tag{16}$$

[4]Navier says that (12) is usually called "equations indéfinies". [26, p.384,389]

where X, Y and Z are positive values.

Next, we get also X', Y' and Z' from the (14) : we suppose that :[5]

- $db'dc' \;\to\; ds\cos l,\;\; l$: the angles by which the tangent plane makes on the surface frame with the plane bc,
- $da'dc' \;\to\; ds\cos m,\;\; m$: samely, the angles with the plane ac,
- $da'db' \;\to\; ds\cos n,\;\; n$: samely, the angles with the plane ab,
- $\iint db'dc', \iint da'dc', \iint da'db' \;\to\; \int ds$,

then from (14), we get the forces operation on the surface of the elastic solid as follows :

$$(5\text{-}4)_{Ne}\quad \begin{cases} X' = \varepsilon\Big[\cos\; l\Big(3\frac{dx'}{da'}+\frac{dy'}{db'}+\frac{dz'}{dc'}\Big)+\cos\; m\Big(\frac{dx'}{db'}+\frac{dy'}{da'}\Big)+\cos\; n\Big(\frac{dx'}{dc'}+\frac{dz'}{da'}\Big)\Big], \\ Y' = \varepsilon\Big[\cos\; l\Big(\frac{dx'}{db'}+\frac{dy'}{da'}\Big)+\cos\; m\Big(\frac{dx'}{da'}+3\frac{dy'}{db'}+\frac{dz'}{dc'}\Big)+\cos\; n\Big(\frac{dy'}{dc'}+\frac{dz'}{db'}\Big)\Big], \\ Z' = \varepsilon\Big[\cos\; l\Big(\frac{dx'}{dc'}+\frac{dz'}{da'}\Big)+\cos\; m\Big(\frac{dy'}{dc'}+\frac{dz'}{db'}\Big)+\cos\; n\Big(\frac{dx'}{da'}+\frac{dy'}{db'}+3\frac{dz'}{dc'}\Big)\Big]. \end{cases} \tag{17}$$

$$\Rightarrow \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \varepsilon \begin{bmatrix} 3\frac{dx'}{da'}+\frac{dy'}{db'}+\frac{dz'}{dc'} & \frac{dx'}{db'}+\frac{dy'}{da'} & \frac{dx'}{dc'}+\frac{dz'}{da'} \\ \frac{dx'}{db'}+\frac{dy'}{da'} & \frac{dx'}{da'}+3\frac{dy'}{db'}+\frac{dz'}{dc'} & \frac{dy'}{dc'}+\frac{dz'}{db'}, \\ \frac{dx'}{dc'}+\frac{dz'}{da'} & \frac{dy'}{dc'}+\frac{dz'}{db'} & \frac{dx'}{da'}+\frac{dy'}{db'}+3\frac{dz'}{dc'} \end{bmatrix} \begin{bmatrix} \cos l \\ \cos m \\ \cos n \end{bmatrix} \tag{18}$$

By the way, when we rearrange (9) to compare with equations of equilibrium in fluid, then (9) becomes (19) as follows :

$$\begin{aligned} &(5\text{-}1)'_{Ne}\quad 0 \\ &= \varepsilon\iiint dadbdc \begin{cases} \Big(3\frac{dx}{da}\frac{\delta dx}{da}+\frac{dx}{db}\frac{\delta dx}{db}+\frac{dx}{dc}\frac{\delta dx}{dc}\Big)+\Big(\frac{dy}{db}\frac{\delta dx}{da}+\frac{dy}{da}\frac{\delta dx}{db}\Big)+\Big(\frac{dz}{dc}\frac{\delta dx}{da}+\frac{dz}{da}\frac{\delta dx}{dc}\Big) \\ \Big(\frac{dx}{da}\frac{\delta dy}{db}+\frac{dx}{db}\frac{\delta dy}{da}\Big)+\Big(\frac{dy}{da}\frac{\delta dy}{da}+3\frac{dy}{db}\frac{\delta dy}{db}+\frac{dy}{dc}\frac{\delta dy}{dc}\Big)+\Big(\frac{dz}{db}\frac{\delta dy}{dc}+\frac{dz}{dc}\frac{\delta dy}{db}\Big) \\ \Big(\frac{dx}{da}\frac{\delta dz}{dc}+\frac{dx}{dc}\frac{\delta dz}{da}\Big)+\Big(\frac{dy}{db}\frac{\delta dz}{dc}+\frac{dy}{dc}\frac{\delta dz}{db}\Big)+\Big(\frac{dz}{da}\frac{\delta dz}{da}+\frac{dz}{db}\frac{\delta dz}{db}+3\frac{dz}{dc}\frac{\delta dz}{dc}\Big) \end{cases} \\ &- \iiint dadbdc(X\delta x+Y\delta y+Z\delta z)-\int ds(X'\delta x'+Y'\delta y'+Z'\delta z'). \end{aligned} \tag{19}$$

Navier deduces the equations of equilibrium in fluid as follows :

$$\begin{aligned} &(3\text{-}24)_{Nf}\quad 0 \\ &= \iiint dxdydz \begin{cases} \big[P-\frac{dp}{dx}-\rho\Big(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\Big)\big]\delta u \\ \big[Q-\frac{dp}{dy}-\rho\Big(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\Big)\big]\delta v \\ \big[R-\frac{dp}{dz}-\rho\Big(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\Big)\big]\delta w \end{cases} \\ &- \varepsilon\iiint dxdydz \begin{cases} \Big(3\frac{du}{dx}\frac{\delta du}{dx}+\frac{du}{dy}\frac{\delta du}{dy}+\frac{du}{dz}\frac{\delta du}{dz}\Big)+\Big(\frac{dv}{dy}\frac{\delta du}{dx}+\frac{dv}{dx}\frac{\delta du}{dy}\Big)+\Big(\frac{dw}{dz}\frac{\delta du}{dx}+\frac{dw}{dx}\frac{\delta du}{dz}\Big) \\ \Big(\frac{du}{dx}\frac{\delta dv}{dy}+\frac{du}{dy}\frac{\delta dv}{dx}\Big)+\Big(\frac{dv}{dx}\frac{\delta dv}{dx}+3\frac{dv}{dy}\frac{\delta dv}{dy}+\frac{dv}{dz}\frac{\delta dv}{dz}\Big)+\Big(\frac{dw}{dy}\frac{\delta dv}{dz}+\frac{dw}{dz}\frac{\delta dv}{dy}\Big) \\ \Big(\frac{du}{dx}\frac{\delta dw}{dz}+\frac{du}{dz}\frac{\delta dw}{dx}\Big)+\Big(\frac{dv}{dy}\frac{\delta dw}{dz}+\frac{dv}{dz}\frac{\delta dw}{dy}\Big)+\Big(\frac{dw}{dx}\frac{\delta dw}{dx}+\frac{dw}{dy}\frac{\delta dw}{dy}+3\frac{dw}{dz}\frac{\delta dw}{dz}\Big) \end{cases} \\ &+ \mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w). \end{aligned} \tag{20}$$

When we compare only the terms of ε between (19) in elastic solid and (20) in fluid, the defference is none, and the both tensor are symmetric respectively.

• ¶ 6. Navier computes the acceleration around the point M. Π is *dencity* of the solid per volume, g is *acceleration of gravity*, then

$$(6\text{-}1)_{Ne}\quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\Big(3\frac{d^2x}{da^2}+\frac{d^2x}{db^2}+\frac{d^2x}{dc^2}+2\frac{d^2y}{dbda}+2\frac{d^2z}{dcda}\Big), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\Big(\frac{d^2y}{da^2}+3\frac{d^2y}{db^2}+\frac{d^2y}{dc^2}+2\frac{d^2x}{dadb}+2\frac{d^2z}{dcdb}\Big), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\Big(\frac{d^2z}{da^2}+\frac{d^2z}{db^2}+3\frac{d^2z}{dc^2}+2\frac{d^2x}{dadc}+2\frac{d^2y}{dbdc}\Big) \end{cases} \tag{21}$$

Poisson comments that ε in (16) and (21) equal to Poisson's corresponding parameter in $(6)_{Pe}$ (= (61)), namely Navier's ε is equivalent to Poison's $\frac{a^2}{2}$, however Navier denies it.

[5] On this method Navier cites Lagrange ([18, pp.113-188,1 partie ,§ 5]), *Solution de différents problèmes de statique*. In fluid case, Navier rethinks this method afterward. c.f. (47).

TABLE 7. Combination between V and δV

	α^2	$\alpha\beta$	$\alpha\gamma$	$\beta\alpha$	β^2	$\beta\gamma$	$\gamma\alpha$	$\gamma\beta$	γ^2
α^2	1				5				7
$\alpha\beta$		2		4					
$\alpha\gamma$			3				6		
$\beta\alpha$		9		10					
β^2	8				11				14
$\beta\gamma$						12		13	
$\gamma\alpha$			16				19		
$\gamma\beta$						18		20	
γ^2	15				17				21

3.1.3. Deduction of the expressions of forces of the molecular action which is under the state of motion.

Navier deduces the expressions of forces of the molecular action which is under the state of motion as follows : [6]

We consider the two molecules M and M'. x, y, z are the values of the rectangular coordinates of M and $x+\alpha, y+\beta, z+\gamma$ are the values of the rectangular coordinates of M'. The length of a rayon emitting from M : $\rho=\sqrt{\alpha^2+\beta^2+\gamma^2}$. The velocities of the molecule M are u, v, w and that of the molecules M' are

$$(3\text{-}3)_{Nf}\quad u+\frac{du}{dx}\alpha+\frac{du}{dy}\beta+\frac{du}{dz}\gamma,\quad v+\frac{dv}{dx}\alpha+\frac{dv}{dy}\beta+\frac{dv}{dz}\gamma,\quad w+\frac{dw}{dx}\alpha+\frac{dw}{dy}\beta+\frac{dw}{dz}\gamma \tag{22}$$

V is the quantity on which the proportional action depends as follows :

$$(3\text{-}4)_{Nf}\quad V=\frac{\alpha}{\rho}\left(\frac{du}{dx}\alpha+\frac{du}{dy}\beta+\frac{du}{dz}\gamma\right)+\frac{\beta}{\rho}\left(\frac{dv}{dx}\alpha+\frac{dv}{dy}\beta+\frac{dv}{dz}\gamma\right)+\frac{\gamma}{\rho}\left(\frac{dw}{dx}\alpha+\frac{dw}{dy}\beta+\frac{dw}{dz}\gamma\right). \tag{23}$$

V represents the force which exists between two certain molecules of fluid. The increment of V is as follows :

$$(3\text{-}5)_{Nf}\quad \delta V=\frac{\alpha}{\rho}\left(\frac{\delta du}{dx}\alpha+\frac{\delta du}{dy}\beta+\frac{\delta du}{dz}\gamma\right)+\frac{\beta}{\rho}\left(\frac{\delta dv}{dx}\alpha+\frac{\delta dv}{dy}\beta+\frac{\delta dv}{dz}\gamma\right)+\frac{\gamma}{\rho}\left(\frac{\delta dw}{dx}\alpha+\frac{\delta dw}{dy}\beta+\frac{\delta dw}{dz}\gamma\right). \tag{24}$$

$f(\rho)$ is a function depends on the distance ρ between M and M'. We define that ψ is the angle of the rayon ρ with its projection on the $\alpha\beta$-plane and φ is the angle which this projection forms with the α axis, and then

$$(3\text{-}9)_{Nf}\quad \alpha=\rho\cos\psi\cos\varphi,\quad \beta=\rho\cos\psi\sin\varphi,\quad \gamma=\rho\sin\psi \tag{25}$$

We calculate $d\rho d\psi d\varphi\rho^2\cos\varphi$ of the element of the volume in the new system of coordinates : (α,β,γ), and integrate with respect to φ, ψ from 0 to $\frac{\pi}{2}$ and with respect to ρ from 0 to ∞.

$$\begin{aligned}(3\text{-}6)_{Nf}\quad &\frac{1}{\rho^4}f(\rho)V\delta V=\\ \frac{f(\rho)}{\rho^4}\quad &\left[\ \alpha\left(\frac{du}{dx}\alpha+\frac{du}{dy}\beta+\frac{du}{dz}\gamma\right)+\beta\left(\frac{dv}{dx}\alpha+\frac{dv}{dy}\beta+\frac{dv}{dz}\gamma\right)+\gamma\left(\frac{dw}{dx}\alpha+\frac{dw}{dy}\beta+\frac{dw}{dz}\gamma\right)\ \right]\times\\ &\left[\ \alpha\left(\frac{\delta du}{dx}\alpha+\frac{\delta du}{dy}\beta+\frac{\delta du}{dz}\gamma\right)+\beta\left(\frac{\delta dv}{dx}\alpha+\frac{\delta dv}{dy}\beta+\frac{\delta dv}{dz}\gamma\right)+\gamma\left(\frac{\delta dw}{dx}\alpha+\frac{\delta dw}{dy}\beta+\frac{\delta dw}{dz}\gamma\right)\ \right],\end{aligned} \tag{26}$$

here, by the symmetry we supposed, we get the relations as follows :

$$\left|\alpha\frac{du}{dy}\beta\right|=\left|\beta\frac{dv}{dx}\alpha\right|,\quad \left|\beta\frac{dv}{dz}\gamma\right|=\left|\gamma\frac{dw}{dy}\beta\right|,\quad \left|\alpha\frac{du}{dz}\gamma\right|=\left|\gamma\frac{dw}{dx}\alpha\right|,$$
$$\left|\alpha\frac{\delta du}{dy}\beta\right|=\left|\beta\frac{\delta dv}{dx}\alpha\right|,\quad \left|\beta\frac{\delta dv}{dz}\gamma\right|=\left|\gamma\frac{\delta dw}{dy}\beta\right|,\quad \left|\alpha\frac{\delta du}{dz}\gamma\right|=\left|\gamma\frac{\delta dw}{dx}\alpha\right|.$$

Because we integrate only $\frac{1}{8}$ volume of the total sphere, we must multiply it by 8.

[6] Navier ([26, pp.399-405])

$(3\text{-}7)_{Nf}$

$$
\begin{aligned}
&8\frac{f(\rho)}{\rho^4}\Big\{\Big(\frac{du}{dx}\frac{\delta du}{dx}\alpha^4+\frac{du}{dy}\frac{\delta du}{dy}\alpha^2\beta^2+\frac{du}{dz}\frac{\delta du}{dz}\alpha^2\gamma^2\Big)+\Big(\frac{dv}{dx}\frac{\delta du}{dy}+\frac{dv}{dy}\frac{\delta du}{dx}\Big)\alpha^2\beta^2+\Big(\frac{dw}{dx}\frac{\delta du}{dz}+\frac{dw}{dz}\frac{\delta du}{dx}\Big)\alpha^2\gamma^2\\
&+\Big(\frac{du}{dx}\frac{\delta dv}{dy}+\frac{du}{dy}\frac{\delta dv}{dx}\Big)\alpha^2\beta^2+\Big(\frac{dv}{dx}\frac{\delta dv}{dx}\alpha^2\beta^2+\frac{dv}{dy}\frac{\delta dv}{dy}\beta^4+\frac{dv}{dz}\frac{\delta dv}{dz}\beta^2\gamma^2\Big)+\Big(\frac{dw}{dy}\frac{\delta dv}{dz}+\frac{dw}{dz}\frac{\delta dv}{dy}\Big)\beta^2\gamma^2\\
&+\Big(\frac{du}{dx}\frac{\delta dw}{dz}+\frac{du}{dz}\frac{\delta dw}{dx}\Big)\alpha^2\gamma^2+\Big(\frac{dv}{dy}\frac{\delta dw}{dz}+\frac{dv}{dz}\frac{\delta dw}{dy}\Big)\beta^2\gamma^2+\Big(\frac{dw}{dx}\frac{\delta dw}{dx}\alpha^2\gamma^2+\frac{dw}{dy}\frac{\delta dw}{dy}\beta^2\gamma^2+\frac{dw}{dz}\frac{\delta dw}{dz}\gamma^4\Big)\Big\} \quad (27)
\end{aligned}
$$

We get 21 terms in (27) from (26). We show the combination between V and δV in Table 6, in which the row is V and the column is δV and the numbers are the order of the description of the 21 terms in (27). By the formulae of the original function on infinite integral :

$$
\begin{cases}
\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x, \quad \int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x,\\
\int \sin^3 x dx = -\frac{1}{3}\cos x(\sin^2 x + 2), \quad \int \cos^3 x dx = \frac{1}{3}\sin x(\cos^2 x + 2),\\
\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x dx, \quad \int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx,\\
\int \sin x \cos^m x dx = -\frac{\cos^{m+1} x}{m+1}, \quad \int \sin^m x \cos x dx = \frac{\sin^{m+1} x}{m+1}
\end{cases}
$$

We get the result of the integration excepting for $\int_0^\infty d\rho$ as follows :,

$$
\begin{aligned}
\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \alpha^4 \cos\psi &= \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^5\psi\cos^4\varphi = \frac{\pi}{10},\\
\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \beta^4 \cos\psi &= \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^5\psi\sin^4\varphi = \frac{\pi}{10},\\
\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \gamma^4 \cos\psi &= \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos\psi\sin^4\varphi = \frac{\pi}{10},\\
\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \alpha^2\beta^2 \cos\psi &= \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^5\psi\sin^2\varphi\cos^2\varphi = \frac{\pi}{30},\\
\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \alpha^2\gamma^2 \cos\psi &= \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^3\psi\sin^2\varphi\cos^2\varphi = \frac{\pi}{30},\\
\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \beta^2\gamma^2 \cos\psi &= \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^3\psi\sin^2\varphi\sin^2\varphi = \frac{\pi}{30}.
\end{aligned}
$$

Total of the sphere is multiplied by 8 taking ε as the common factor :

$$(3\text{-}10)_{Nf} \quad \varepsilon \equiv \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) = \frac{4\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho) \tag{28}$$

We get now ε of (2), and using (43) and (44) it turns out the term of $\frac{\varepsilon}{\rho}\Delta\mathbf{u}$ of the today's formulation (1) from next :

$$
\varepsilon\begin{cases}
3\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+2\frac{dudv}{dxdy}+2\frac{dwdu}{dzdx},\\
\frac{d^2v}{dx^2}+3\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+2\frac{dudv}{dxdy}+2\frac{dudw}{dydz},\\
\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+3\frac{d^2w}{dz^2}+2\frac{dvdw}{dydz}+2\frac{dwdu}{dzdx}
\end{cases}
\quad\Rightarrow\quad \varepsilon\Delta\mathbf{u}.
$$

Exactly speaking, Navier ([26, p.405]) says this ε must be mupltiply by $\frac{1}{2}$, for double count, when we get the total moments of the forces caused by the reciprocal actions of the molecules of a fluid in the following section, as follows :

$$(3\text{-}9)_{Ne} \qquad \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \tag{29}$$

By this reason, Darrigol cites Navier's tensor from this in using tensor notation.[7]

[7]O.Darrigol [11, p.112] interprets that this is Navier's tensor as follows :

$$\frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho) \equiv k, \quad M = \int \sigma_{ij}\partial_i w_j d\tau,$$
$$\sigma_{ij} = -kN^2(\delta_{ij}\partial_k u_k + \partial_i u_j + \partial_j u_i) \equiv -kN^2(\delta_{ij}u_{kk} + u_{ji} + u_{ij}),$$
where $N = 1$.

3.1.4. Deduction of the expressions of the total moments of the forces caused by the reciprocal actions of the molecules of a fluid.

Navier uses the above results to seek the expression of the total moments of the forces caused by the reciprocal actions of the molecules of a fluid as follows : [8] Here, we rotate the rectangular coordinates for γ' to coincide with the direction of a rayon MN of which M is the common origin of the both rectangular coodinates of α, β, γ and α', β', γ' satisfying $\varphi = r$ and $\psi = s$ and then we get the new relation of α', β' and γ' from (25) as follows :

$$\alpha' = \rho \cos\psi \cos\varphi = \rho \cos r \cos s, \quad \beta' = \rho \cos\psi \sin\varphi = \rho \cos r \sin s, \quad \gamma' = \rho \sin\psi = \rho \sin r \qquad (30)$$

In fig.1, we suppose that : the point P is the projected point on $\alpha\beta$-plane from N. The angle of PMN equals to s. N, R and Q are on the common line on the $\beta'\gamma'$-plane, and plane MNR and plane MRQ are on the common $\beta'\gamma'$-plane. $MN \perp MQ$, and $MR \perp MP$. Therefore, the angle made by MQ and MR equals to s.

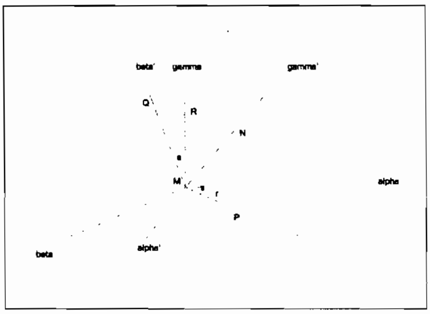

fig.1 Rotation of coordinates

From the above, we get as follows :

$$(3\text{-}17)_{Nf} \quad \begin{cases} \alpha = -\alpha' \sin r + \beta' \cos r \sin s + \gamma' \cos r \cos s, \\ \beta = \ \ \alpha' \cos r + \beta' \sin r \sin s + \gamma' \sin r \cos s, \\ \gamma = \qquad\qquad \beta' \cos s \qquad\qquad - \gamma' \sin s \end{cases}$$

$$or \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\sin r & \cos r \sin s & \cos r \cos s \\ \cos r & \sin r \sin s & \sin r \cos s \\ 0 & \cos s & -\sin s \end{bmatrix} \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix} \equiv A \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix},$$

where each last terms of the right hand-side of α, β, γ (or the values in the 3rd column of the 3×3 matrix for the transformation) are the original values of (25) excepting for the term of γ', and the rest terms are added by the rotation. By the way, if we call this rotation matrix A, we get $det(A) = 1$, so that $A^{-1} = \tilde{A} = A^T$, i.e.

$$\begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix} = A^{-1} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\sin r & \cos r & 0 \\ \cos r \cos s & \sin r \sin s & \cos s \\ \cos r \cos s & \sin r \cos s & -\sin s \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

As well as (26) using (22),(23) and (24), it turns out the expression which must be integrate for all the value of α' and β' and for the positive value only of γ'. We get as follows :

$$(3\text{-}18)_{Nf} \quad \frac{1}{\rho^2} F(\rho) V \delta V = \frac{F(\rho)}{\rho^2} \times$$
$$[\ \alpha'(-u \sin r + v \cos r) + \beta'(u \cos r \sin s + v \sin r \sin s + w \cos s) + \gamma'(u \cos r \cos s + v \sin r \cos s - w \sin s)\ ] \times$$
$$[\ \alpha'(-\delta u \sin r + \delta v \cos r) + \beta'(\delta u \cos r \sin s + \delta v \sin r \sin s + \delta w \cos s) + \gamma'(\delta u \cos r \cos s + \delta v \sin r \cos s - \delta w \sin s$$

(3

In analogy with Lagrange's reasoning, Navier then integrated by parts to get

$$M = \oint \sigma_{ij} \partial_i w_j dS_i - \int (\partial_i \sigma_{ij}) w_j d\tau.$$

[8] Navier ([26, pp.405-416])

In the right-hand side of (31), we get excepting $\frac{F(\rho)}{\rho^2}$ if we put as follows :

$$[\alpha' a+\beta' b+\gamma' c][\alpha'(d\delta u+e\delta v)+\beta'(f\delta u+g\delta v+h\delta w)+\gamma'(i\delta u+j\delta v+k\delta w)]$$

or

$$\left(\left[\begin{array}{lll} a & b & c \end{array}\right]\left[\begin{array}{c}\alpha'\\ \beta'\\ \gamma'\end{array}\right]\right)\left(\left[\begin{array}{lll}\alpha' & \beta' & \gamma'\end{array}\right]\left(\left[\begin{array}{ccc} d & e & 0\\ f & g & h\\ i & j & k\end{array}\right]\left[\begin{array}{c}\delta u\\ \delta v\\ \delta w\end{array}\right]\right)\right),$$

then we get

$$\left[\begin{array}{c}\alpha'\beta'\\ \beta'\gamma'\\ \gamma'\alpha'\end{array}\right]=\left[\begin{array}{lll} fa+db & ga+eb & ha\\ fc+ib & gc+jb & hc+kb\\ ia+cd & ja+ce & ka\end{array}\right]\left[\begin{array}{c}\delta u\\ \delta v\\ \delta w\end{array}\right].$$

We get effectively as follows :

$$\begin{aligned}\alpha'\beta' = &\Big\{-2u\sin r\cos r\sin s+v\sin s(\cos^2 r-\sin^2 r)-w\sin r\cos s\Big\}\delta u+\\ &\Big\{\ 2v\sin r\cos r\sin s+u\sin s(\cos^2 r-\sin^2 r)+w\cos r\cos s\Big\}\delta v+\\ &\Big\{\cos s(v\cos r-u\sin r)\Big\}\delta w\end{aligned}$$

$$\begin{aligned}\beta'\gamma' = &\cos r\Big\{2u\cos r\sin s\cos s+2v\sin r\sin s\cos s+w(\cos^2 s-\sin^2 s)\Big\}\delta u+\\ &\sin r\Big\{2u\cos r\sin s\cos s+2v\sin r\sin s\cos s+w(\cos^2 s-\sin^2 s)\Big\}\delta v+\\ &\Big\{u\cos r+v\sin r-2w\sin s\cos s\Big\}\delta w\end{aligned}$$

$$\begin{aligned}\gamma'\alpha' = &\Big\{-2u\sin r\cos r\cos s+v\cos s(\cos^2 r-\sin^2 r)+w\sin s\sin r\Big\}\delta u+\\ &\Big\{\ 2v\sin r\cos r\cos s+u\cos(\cos^2 r-\sin^2 r)-w\sin s\cos r\Big\}\delta v+\\ &\Big\{\sin s(u\sin r-v\cos r)\Big\}\delta w\end{aligned}$$

On this point, Navier explains as follows :

> On the above expression, we must integrate for all the value with respect to α' and β', but with respect to γ', only positive value. This operation becomes simple giving consideration to
>
> - that if we consider four points placed symmetrically, this sign for γ' is positive, but the other coordinates α' and β' differ from each other by sign two by two, and
> - that we add the values which the above expression (31) takes in these four points, it rests, as the result of the addition, only the terms which relate to the terms of α' to the power of 2 and the terms of β' to the power of 2, the terms appear by the multiplication with 4.
>
> Hence performing the multi indexes, all is reduced to integrate the quality in the volume of $\frac{1}{8}$ of a sphere where α', β' and γ' take the positive values as follows :

$$(3\text{-}19)_{Nf}\quad 4\frac{F(\rho)}{\rho^2}\left\{\begin{array}{ll}\alpha'^2 & \left\{\begin{array}{l}(u\sin^2 r-v\sin r\cos r)\delta u\\ (-u\sin r\cos r+v\cos^2 r)\delta v\end{array}\right\}\\ \beta'^2 & \left\{\begin{array}{l}(u\cos^2 r\sin^2 s+v\sin r\cos r\sin^2 s+w\cos r\sin s\cos s)\delta u\\ (u\sin r\cos r\sin^2 s+v\sin^2 r\sin^2 s+w\sin r\sin s\cos s)\delta v\\ (u\cos r\sin s\cos s+v\sin r\sin s\cos s+w\cos^2 s\qquad)\delta w\end{array}\right\}\\ \gamma'^2 & \left\{\begin{array}{l}(u\cos^2 r\cos^2 s+v\sin r\cos r\cos^2 s-w\cos r\sin s\cos s)\delta u\\ (u\sin r\cos r\cos^2 s+v\sin^2 r\cos^2 s-w\sin r\sin s\cos s)\delta v\\ (-u\cos r\sin s\cos s-v\sin r\sin s\cos s+w\sin^2 s\qquad)\delta w\end{array}\right\}\end{array}\right. \qquad (32)$$

$$\alpha'=\rho\cos\psi\cos\varphi,\quad \beta'=\rho\cos\psi\sin\varphi,\quad \gamma'=\rho\sin\psi$$

For we calculate the element of volume $d\rho d\psi d\varphi\rho^2\cos\psi$ with respect to ψ and φ from 0 to $\frac{\pi}{2}$, we get the following three results of the finite integrations :

$$\frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\alpha'^2\cos\psi = \frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\cos^3\psi\cos^2\varphi = \frac{\pi}{6},$$
$$\frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\beta'^2\cos\psi = \frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\cos^3\psi\sin^2\varphi = \frac{\pi}{6},$$
$$\frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\gamma'^2\cos\psi = \frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\sin^2\psi\cos\psi = \frac{\pi}{6}$$

$F(\rho)$ is the same function as $f(\rho)$ in (26), which is a function which depends on the distance ρ between M and M'. Taking $\frac{\pi}{6}$ as the common factor, then we put

$$(3\text{-}22)_{Nf} \quad \frac{4\pi}{6}\int_0^\infty d\rho\rho^2 F(\rho) = \frac{2\pi}{3}\int_0^\infty d\rho\rho^2 F(\rho) \equiv E. \qquad (33)$$

and define :

$$(3\text{-}23)_{Nf} \quad E(u\delta u + v\delta v + w\delta w)$$

for the expression which we seek for the sum of the moments of the total actions which caused between the molecules of the wall and the fluid, following the direction which pass by the point of the separation of the fluid and the wall and the fluid, and which can be regarded as the measure of its reciprocal action. We get the following equilibrium of a fluid using ε of (28) and the above $E(u\delta u + v\delta v + w\delta w)$:

$$\begin{aligned}(3\text{-}24)_{Nf} \quad 0 =& \iiint dxdydz\begin{cases}\left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\right]\delta u\\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\right]\delta v\\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\right]\delta w\end{cases}\\ &- \varepsilon\iiint dxdydz\begin{cases}3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz} + \frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy} + \frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\\ \frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx} + \frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz} + \frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\\ \frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx} + \frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy} + \frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\end{cases}\\ &+ \mathbf{S}ds^2E(u\delta u + v\delta v + w\delta w). \qquad (34)\end{aligned}$$

Here, **S** means the integration in the total surface of the fluid, in varing the quantity E of (33), following the nature of the solid with which this surface is in contact. Shifting d to the front of δ of the middle term of the right hand-side of (34) and by Taylor expansion using the partial integral

$(3\text{-}25)_{Nf}$

$$\varepsilon \iiint dxdydz \begin{cases} \left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dzdx}\right)\delta u \\ \left(2\frac{d^2u}{dxdy} + \frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2w}{dydz}\right)\delta v \\ \left(2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2}\right)\delta w \end{cases} \tag{35}$$

$$+ \quad \varepsilon \iint dy'dz'\left[\left(3\frac{du'}{dx'} + \frac{dv'}{dy'} + \frac{dw'}{dz'}\right)\delta u' + \left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta v' + \left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta w'\right] \tag{36}$$

$$+ \quad \varepsilon \iint dx'dz'\left[\left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta u' + \left(\frac{du'}{dx'} + 3\frac{dv'}{dy'} + \frac{dw'}{dz'}\right)\delta v' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta w'\right] \tag{37}$$

$$+ \quad \varepsilon \iint dx'dy'\left[\left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta u' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta v' + \left(\frac{du'}{dx'} + \frac{dv'}{dy'} + 3\frac{dw'}{dz'}\right)\delta w'\right] \tag{38}$$

$$- \quad \varepsilon \iint dy''dz''\left[\left(3\frac{du''}{dx''} + \frac{dv''}{dy''} + \frac{dw''}{dz''}\right)\delta u'' + \left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta v'' + \left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta w''\right] \tag{39}$$

$$- \quad \varepsilon \iint dx''dz''\left[\left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta u'' + \left(\frac{du''}{dx''} + 3\frac{dv''}{dy''} + \frac{dw''}{dz''}\right)\delta v'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta w''\right] \tag{40}$$

$$- \quad \varepsilon \iint dx''dy''\left[\left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta u'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta v'' + \left(\frac{du''}{dx''} + \frac{dv''}{dy''} + 3\frac{dw''}{dz''}\right)\delta w''\right] \tag{41}$$

By the way, if we check the ε terms of $\delta u', \delta v', \delta w'$, after replacing $\mathbf{u} = \{u, v, w\}$ of fluid $\Leftrightarrow$ $\{x, y, z\}$ of elastic solid, and the coordinate system : $\{x, y, z\}$ of fluid $\Leftrightarrow$ $\{a, b, c\}$ of elastic solid, then we can see the coincidence with the tensor between the equation (15) or (42) in elastic solid and (36)-(38) in fluid as follows :

$$\Rightarrow \int ds \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \varepsilon \begin{bmatrix} 3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{dc'} + \frac{dz'}{da'} \\ \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dy'}{dc'} + \frac{dz'}{db'}, \\ \frac{dx'}{dc'} + \frac{dz'}{da'} & \frac{dy'}{dc'} + \frac{dz'}{db'} & \frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'} \end{bmatrix} \begin{bmatrix} \iint db'\,dc' \\ \iint da'dc' \\ \iint da'db' \end{bmatrix}, \tag{42}$$

here this tensor is symmetric.

Using the following equations deduced from the conservation law :

$$(3\text{-}26)_{Nf} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{43}$$

and

$$(3\text{-}27)_{Nf}) \quad \begin{cases} \frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{d^2u}{dx^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz} = 0, \\ \frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{d^2v}{dy^2} + \frac{d^2u}{dxdy} + \frac{d^2w}{dydz} = 0, \\ \frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{d^2w}{dz^2} + \frac{d^2u}{dxdz} + \frac{d^2v}{dydz} = 0, \end{cases} \tag{44}$$

we get the short expression as follows :

$$\begin{aligned} (3\text{-}28)_{Nf} \quad & \varepsilon \iiint dxdydz\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)\delta u + \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)\delta v + \left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)\delta w] \\ + \quad & \varepsilon \iint dy'dz'\left[2\frac{du'}{dx'}\delta u' + \left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta v' + \left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta w'\right] \\ + \quad & \varepsilon \iint dx'dz'\left[\left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta u' + 2\frac{dv'}{dy'}\delta v' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta w'\right] \\ + \quad & \varepsilon \iint dx'dy'\left[\left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta u' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta v' + 2\frac{dw'}{dz'}\delta w'\right] \\ - \quad & \varepsilon \iint dy''dz''\left[2\frac{du''}{dx''}\delta u'' + \left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta v'' + \left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta w''\right] \\ - \quad & \varepsilon \iint dx''dz''\left[\left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta u'' + 2\frac{dv''}{dy''}\delta v'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta w''\right] \\ - \quad & \varepsilon \iint dx''dy''\left[\left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta u'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta v'' + 2\frac{dw''}{dz''}\delta w''\right] \end{aligned} \tag{45}$$

Considering $\mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w)$ of (34) and the total of the rest terms of the (45) is zero, we get the last expression from (34) and the first term of (45) as follows :

$$(3\text{-}29)_{Nf}\quad 0=\iiint dxdydz\begin{cases}\left[P-\frac{dp}{dx}-\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)+\varepsilon\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)\right]\delta u\\ \left[Q-\frac{dp}{dy}-\rho\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)+\varepsilon\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\right]\delta v\\ \left[R-\frac{dp}{dz}-\rho\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right)+\varepsilon\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)\right]\delta w\end{cases}\quad (46)$$

At last, we get (2) from (46).

On the other hand, to deduce (47) from (45), we transpose (36)-(38) as follows :

$$(3\text{-}30)_{Nf}\quad\begin{cases}Eu\delta u+\varepsilon\left[2\iint dy'dz'\frac{du'}{dx'}+\iint dx'dz'\left(\frac{du'}{dy'}+\frac{dv'}{dx'}\right)+\iint dx'dy'\left(\frac{du'}{dz'}+\frac{dw'}{dx'}\right)\right]\delta u'+\cdots\delta u''+\cdots=0\\ Ev\delta v+\varepsilon\left[\iint dy'dz'\left(\frac{du'}{dy'}+\frac{dv'}{dx'}\right)+2\iint dx'dz'\frac{dv'}{dy'}+\iint dx'dy'\left(\frac{dv'}{dz'}+\frac{dw'}{dy'}\right)\right]\delta v'+\cdots\delta v''+\cdots=0\\ Ew\delta w+\varepsilon\left[\iint dy'dz'\left(\frac{du'}{dz'}+\frac{dw'}{dx'}\right)+\iint dx'dz'\left(\frac{dv'}{dz'}+\frac{dw'}{dy'}\right)+2\iint dx'dy'\frac{dw'}{dz'}\right]\delta w'+\cdots\delta w''+\cdots=0.\end{cases}$$

3.1.5. Boundary condition.

About $\mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w)$ of (34) and the total of the rest terms of (45), Navier explains as follows : regarding the conditions which react at the points of the surface of the fluid, if we substitute

- $dydz \quad\rightarrow\quad ds^2\cos l,\quad l$: the angles by which the tangent plane makes on the surface frame with the plane yz,
- $dxdz \quad\rightarrow\quad ds^2\cos m,\quad m$: samely, the angles with the plane xz,
- $dxdy \quad\rightarrow\quad ds^2\cos n,\quad n$: samely, the angles with the plane xy,
- $\iint dydz, \iint dxdz, \iint dxdy \quad\rightarrow\quad \mathbf{S}ds^2$,

then because the affected terms by the quantities $\delta u, \delta v$ and δw respectively reduce to zero, the following determinated equations should hold for any points of the surface of the fluid :

$$(3\text{-}32)_{Nf}\quad\begin{cases}Eu+\varepsilon\left[\cos l\,2\frac{du}{dx}+\cos m\left(\frac{du}{dy}+\frac{dv}{dx}\right)+\cos n\left(\frac{du}{dz}+\frac{dw}{dx}\right)\right]=0,\\ Ev+\varepsilon\left[\cos l\left(\frac{du}{dy}+\frac{dv}{dx}\right)+\cos m\,2\frac{dv}{dy}+\cos n\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\right]=0,\\ Ew+\varepsilon\left[\cos l\left(\frac{du}{dz}+\frac{dw}{dx}\right)+\cos m\left(\frac{dv}{dz}+\frac{dw}{dy}\right)+\cos n\,2\frac{dw}{dz}\right]=0,\end{cases}\quad (47)$$

here the value of the constant E which varifies following the nature of the solid with which the fluid is in contact. (47) express the boundary condition. The first terms of the left-hand side of (47) are defined by (33) for the expression which we seek for the sum of the moments of the total actions which caused between the molecules of the boundary and the fluid, and the second terms are the normal derivatives come from (45). Here, (47) is put by :

$$E\begin{bmatrix}u\\ v\\ w\end{bmatrix}+\varepsilon\begin{bmatrix}2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{du}{dz}+\frac{dw}{dx}\\ \frac{du}{dy}+\frac{dv}{dx} & 2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy}\\ \frac{du}{dz}+\frac{dw}{dx} & \frac{dv}{dz}+\frac{dw}{dy} & 2\frac{dw}{dz}\end{bmatrix}\begin{bmatrix}\cos l\\ \cos m\\ \cos n\end{bmatrix}=0\quad (48)$$

If puting the basis of the tensor as $\begin{bmatrix}\cos l & \cos m & \cos n\end{bmatrix}^T$, then the tensor part of (48) is expressed as follows :

$$t_{ij}=-\varepsilon[\{2v_{i,j}-(v_{i,j}+v_{j,i})\}\delta_{ij}+(v_{i,j}+v_{j,i})]=-\varepsilon\{0\delta_{ij}+(v_{i,j}+v_{j,i})\}=-\varepsilon(v_{i,j}+v_{j,i}).\quad (49)$$

Moreover, by using Darrigol's simple notation[9], we can express this condition as

$$E\mathbf{v}+\varepsilon\partial_\perp\mathbf{v}_\parallel=\mathbf{0},$$

where $\partial_\perp$ is the normal derivative, and $\mathbf{v}_\parallel$ is the component of the fluid velocity parallel to the surface.

3.2. Poisson's principle and equations.

[9]Darrigol [11, p.115]

3.2.1. Principle and equations in elastic solid.

We deduce K and k in accordance with Poisson[45, p.368-405, §1-§16] in followings.

- § 2. For abbreviation, we put as follows :

$$\begin{cases} ax_1 + by_1 + c(z_1 - \zeta_1) \equiv \phi, \\ a'x_1 + b'y_1 + c'(z_1 - \zeta_1) \equiv \psi, \\ a''x_1 + b''y_1 + c''(z_1 - \zeta_1) \equiv \theta, \end{cases} \qquad \begin{cases} \phi\frac{du}{dx} + \psi\frac{du}{dy} + \theta\frac{du}{dz} \equiv \phi', \\ \phi\frac{dv}{dx} + \psi\frac{dv}{dy} + \theta\frac{dv}{dz} \equiv \psi', \\ \phi\frac{dw}{dx} + \psi\frac{dw}{dy} + \theta\frac{dw}{dz} \equiv \theta' \end{cases} \tag{50}$$

Namely,

$$\begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 - \zeta \end{bmatrix}, \quad \begin{bmatrix} \phi' \\ \psi' \\ \theta' \end{bmatrix} \equiv \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix} = \nabla\mathbf{u} \cdot \begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix}$$

$$r^2 = \phi^2 + \psi^2 + \theta^2,$$

$$(r')^2 = (\phi + \phi')^2 + (\psi + \psi')^2 + (\theta + \theta')^2,$$

$$r^2 = x_1^2 + y_1^2 + (z_1 - \zeta_1)^2,$$

$$(r')^2 = r^2 + 2\phi\phi' + 2\psi\psi' + 2\theta\theta' + (\phi')^2 + (\psi')^2 + (\theta')^2$$

- § 3. We assume that α : the average molecular interval, ω : surface, $\frac{\omega}{\alpha^2}$: the number of molecules in ω.

$$P = \sum \frac{(\phi + \phi')\zeta}{\alpha^3 r'} fr', \quad Q = \sum \frac{(\psi + \psi')\zeta}{\alpha^3 r'} fr' \quad R = \sum \frac{(\theta + \theta')\zeta}{\alpha^3 r'} fr'. \tag{51}$$

- § 4.

$$r' = r + \frac{1}{r}(\phi\phi' + \psi\psi' + \theta\theta')$$

At the same degree of approximation, we get as follows : [10]

$$\frac{1}{r'} fr' = \frac{1}{r} fr + (\phi\phi' + \psi\psi' + \theta\theta')\frac{d.\frac{1}{r}fr}{rdr}$$

We take from (50) and (51) as follows :[11]

$$(1)_{Pe} \quad \begin{cases} P = \sum \frac{(\phi+\phi')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta')\frac{\phi\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr}, \\ Q = \sum \frac{(\psi+\psi')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta')\frac{\psi\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr}, \\ R = \sum \frac{(\theta+\theta')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta')\frac{\theta\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr}, \end{cases} \tag{52}$$

$$\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \sum \left(\begin{bmatrix} (\phi + \phi') & (\phi\phi' + \psi\psi' + \theta\theta')\phi \\ (\psi + \psi') & (\phi\phi' + \psi\psi' + \theta\theta')\psi \\ (\theta + \theta') & (\phi\phi' + \psi\psi' + \theta\theta')\theta \end{bmatrix} \begin{bmatrix} \frac{\zeta fr}{\alpha^3 r} \\ \frac{\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix} \right)$$

We donate :

β : the angle between the vectoriel rayon of one of molecules : r and the axis of ζ, and

γ : the angle which the projection of the rayon on the $x-y$ plane makes with the axis of x. We have :

$$\begin{cases} x_1 = r\cos\beta\cos\gamma, \\ y_1 = r\sin\beta\sin\gamma, \\ \zeta = r\cos\beta, \end{cases}$$

The quantities which majored under the $\sum$ take the form : pFr, which is expressed by

p : an entire function with sines and cosines of β and γ,

Fr : a same function as fr, of which value are insensible for total sensible value of the variable, and

[10] We correct this equation. Poisson[47], the corresponding equation (67), there is $\frac{1}{rdr}$

[11] We use $_{Pe}$ in the left-side equation number as Poisson's equation number in [45]. And $_{Pf}$ means Poisson[47]

moreover, which is equal to 0 for the paticular value of $r = 0$.
We consider that the summation which is the question is composed by the parties of the form :

$$\sum [(\sum\sum p)Fr],$$

here, the outer $\sum$ corresponds to r and can extend to $r = \infty$, and the inner double $\sum$s correspond to β and γ.

- § 5. The value : $\sum\sum p$ related to sr^2 is assumed by the product of p and the number of molecules which contain in the surface of sr^2, and which is expressed by $\frac{sr^2}{\alpha^2}$. We consider the half surface of the sphere with the radius : $r = 1$ on the $x_1 - y_1$ plane as follows :

$$\frac{r^2}{\alpha^2}\sum\sum ps,$$

This new summation extends to the all parties of the half surface which has the unit for the radius. Because p is not a function of the group of that which decrease very rapidly, we can change s with the diffrential element of the above surface, and the sign : $\sum$ with the signs of integration, we can take as follows :

$$s = \sin\beta d\beta d\gamma, \quad \sum\sum ps = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} p\sin\beta d\beta d\gamma,$$

$$\sum\sum p = \frac{r^2}{\alpha^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} p\sin\beta d\beta d\gamma,$$

$$\sum [(\sum\sum p)Fr] = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} p\sin\beta d\beta d\gamma \sum \frac{r^2}{\alpha^2}Fr.$$

- § 6.

$$\phi = gr, \quad \psi = hr, \quad \theta = lr, \quad \phi' = g'r, \quad \psi' = h'r, \quad \theta' = l'r,$$

$$\begin{cases} g = a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta, & g' = g\frac{du}{dx} + h\frac{du}{dy} + l\frac{du}{dz}, \\ h = a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta, & h' = g\frac{dv}{dx} + h\frac{dv}{dy} + l\frac{dv}{dz}, \\ l = a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta, & l' = g\frac{dw}{dx} + h\frac{dw}{dy} + l\frac{dw}{dz} \end{cases}$$

In brief :

$$\begin{bmatrix} g \\ h \\ l \end{bmatrix} = \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} \sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ -\cos\beta \end{bmatrix}, \quad \begin{bmatrix} g' \\ h' \\ l' \end{bmatrix} = \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} g \\ h \\ l \end{bmatrix} = \nabla\mathbf{u}\cdot \begin{bmatrix} g \\ h \\ l \end{bmatrix}$$

By using the *effective transformation* by Poisson we get from (51) as follows :

$$\begin{cases} P = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[(g+g')\sum \frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')g\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[(h+h')\sum \frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')h\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[(l+l')\sum \frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')l\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases} \tag{53}$$

(53) implies as follows :

$$\begin{aligned} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Delta\left(\begin{bmatrix} g+g' & (gg'+hh'+ll')g \\ h+h' & (gg'+hh'+ll')h \\ l+l' & (gg'+hh'+ll')l \end{bmatrix} \begin{bmatrix} \sum \frac{r^3 fr}{\alpha^5} \\ \sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix} \right), \\ &\equiv \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Delta\left(\begin{bmatrix} g+g' & P' \\ h+h' & Q' \\ l+l' & R' \end{bmatrix} \begin{bmatrix} K \\ k \end{bmatrix} \right), \end{aligned} \tag{54}$$

where

$$\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix} \equiv$$

$$\begin{bmatrix} (g^3\nabla_x u + g^2h\nabla_y u + g^2lg\nabla_z u) + (g^2h\nabla_x v + gh^2\nabla_y v + ghl\nabla_z v) + (g^2l\nabla_x w + ghl\nabla_y w + gl^2\nabla_z w) \\ (g^2h\nabla_x u + gh^2\nabla_y u + ghl\nabla_z u) + (gh^2\nabla_x v + h^3\nabla_y v + h^2l\nabla_z v) + (ghl\nabla_x w + h^2l\nabla_y w + hl^2\nabla_z w) \\ (g^2l\nabla_x u + ghl\nabla_y u + gl^2g\nabla_z u) + (ghl\nabla_x v + h^2l\nabla_y v + hl^2\nabla_z v) + (gl^2\nabla_x w + hl^2\nabla_y w + l^3\nabla_z w) \end{bmatrix} \quad (55)$$

$\Delta := \cos\beta\cdot\sin\beta\, d\beta\, d\gamma$, $\nabla_x u := \frac{du}{dx}$, etc , $K := \sum \frac{r^3 fr}{\alpha^5}$, $k := \sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$.
We use the following integral formulae :

$$\begin{cases} \int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x, \\ \int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x, \\ \int \sin x\cos x dx = \frac{1}{2}\sin^2 x, \\ \int \sin^2 x\cos^2 x dx = -\frac{1}{8}(\frac{1}{4}\sin 4x - x), \\ \int \sin x\cos^m x dx = -\frac{\cos^{m+1}x}{m+1}, \\ \int \sin^m x\cos x dx = \frac{\sin^{m+1}x}{m+1}, \\ \int \cos^m x\sin^n x dx = \frac{\cos^{m-1}x\sin^{n+1}x}{m+n} + \frac{m-1}{m+n}\int\cos^{m-2}x\sin^n x dx, \quad (m>2\ \&\ n>1), \\ \int \cos^m x\sin^n x dx = -\frac{\sin^{n-1}x\cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int\cos^m x\sin^{n-2}x dx, \quad (m>1\ \&\ n>2) \end{cases} \quad (56)$$

We get as follows :

$$\begin{cases} g + g' = (a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta)(1 + \frac{du}{dx}) + h\frac{du}{dy} + l\frac{du}{dz}, \\ h + h' = g\frac{dv}{dx} + (a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta)(1 + \frac{dv}{dy}) + l\frac{dv}{dz}, \\ l + l' = g\frac{dw}{dx} + h\frac{dw}{dy} + (a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta)(1 + \frac{dw}{dz}) \end{cases}$$

We get the integral of $g + g'$ as follows. We put : $A \equiv a\sin\beta\cos\gamma + b\sin\beta\sin\gamma$ and $B \equiv c\cos\beta$.

$$\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} g\Delta = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \big(a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta\big)\cos\beta\sin\beta d\beta \equiv \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} (A - B)\cos\beta\sin\beta d\beta.$$

$$\begin{aligned} \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} A\cos\beta\sin\beta d\beta &= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta\Big(a\sin^2\beta\cos\beta\cos\gamma + b\sin^2\beta\sin\gamma\cos\beta\Big) \\ &= a\Big[\frac{\sin^3\beta}{3}\Big]_0^{2\pi} \int_0^{2\pi} d\gamma\cos\gamma + b\Big[\frac{\sin^3\beta}{3}\Big]_0^{2\pi} \int_0^{2\pi} d\gamma\sin\gamma = 0 \end{aligned}$$

$$\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} -B\cos\beta\sin\beta d\beta = c\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} -\cos^2\beta\sin\beta d\beta = -c\int_0^{2\pi} d\gamma\Big[-\frac{\cos^3\beta}{3}\Big]_0^{\frac{\pi}{2}} = -\frac{2}{3}\pi c$$

We get the following summary of the first half of (54) by the same way as in *above* :

$$\begin{cases} \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(g + g')\Delta = -\frac{2\pi}{3}\Big(c + c\frac{du}{dx} + c'\frac{du}{dy} + c''\frac{du}{dz}\Big), \\ \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(h + h')\Delta = -\frac{2\pi}{3}\Big(c' + c\frac{dv}{dx} + c'\frac{dv}{dy} + c''\frac{dv}{dz}\Big), \\ \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(l + l')\Delta = -\frac{2\pi}{3}\Big(c'' + c\frac{dw}{dx} + c'\frac{dw}{dy} + c''\frac{dw}{dz}\Big), \end{cases} = -\frac{2\pi}{3}(\mathbf{c} + \nabla\mathbf{u}\cdot\mathbf{c}),$$

where $\mathbf{c} = \begin{pmatrix} c & c' & c'' \end{pmatrix}^T$. Here using the relations as follows :

$$a^2 + b^2 + c^2 = 1, \quad aa' + bb' + cc' = 0, \quad a'a'' + b'b'' + c'c'' = 0, \quad a''a + b''b + c''c = 0.$$

3.2.2. Summation of last half term.

We show only the value of $\iint g^3\Delta$ in (55) in detail.

$$\iint g^3\Delta = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} g^3 \cos\beta \sin\beta d\beta$$

$$= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} (a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta)^3 \cos\beta\sin\beta d\beta$$

$$= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta\Big[(a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta)\Big(a^2\sin^2\beta\cos^2\gamma + b^2\sin^2\beta\sin^2\gamma + c^2\cos^2\beta$$

$$+ \quad 2ab\sin^2\beta\cos\gamma\sin\gamma - 2bc\sin\beta\cos\beta\sin\gamma - 2ca\sin\beta\cos\beta\cos\gamma\Big)\cos\beta\sin\beta\Big]$$

When we arrange $\iint g^3\Delta$ in respect to c's terms, then we may compute only 5 c's terms as follows :

$$\Big[(a\sin\beta\cos\gamma) * (-2ca\sin\beta\cos\beta\cos\gamma)$$

$$+ \quad (b\sin\beta\sin\gamma) * (-2bc\sin\beta\cos\beta\sin\gamma)$$

$$- \quad c\cos\beta * \left(a^2\sin^2\beta\cos^2\gamma + b\sin^2\beta\sin^2\gamma + c^2\cos^2\beta\right)\Big]\cos\beta\sin\beta$$

$$= \quad -c\Big(- 2a^2\sin^3\beta\cos^2\beta\cos^2\gamma$$

$$+ \quad b^2\sin^2\gamma\sin^3\beta\sin^2\beta$$

$$+ \quad a^2\sin^3\beta\cos^2\beta\cos^2\gamma + b^2\sin^3\beta\cos^2\beta\sin^2\gamma + c^2\cos^4\beta\sin\beta\Big)$$

$$\equiv \quad -c(A_1 + B_1 + A_2 + B_2 + C)$$

We compute the first term : $-ca^2$'s term of integral in the right-hand side in above as follows :

$-cA_1 : a * (-2ca)$:

$$-2a^2c\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta\sin^3\beta\cos^2\beta\cos^2\gamma = -\frac{4}{15}a^2c\Big[\frac{\gamma}{2} + \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{4}{15}\pi$$

$-cA_2 : -c * a^2$:

$$-ca^2\int_0^{2\pi}\cos^2\gamma d\gamma \int_0^{\frac{\pi}{2}} \sin^3\beta\cos^2\beta d\beta$$

$$= \quad -ca^2\int_0^{2\pi}\cos^2\gamma d\gamma\Big(\Big[-\frac{\sin^2\beta\cos^3\beta}{5}\Big]_0^{\frac{\pi}{2}} + \frac{2}{5}\int_0^{\frac{\pi}{2}}\cos^2\beta\sin\beta d\beta\Big)$$

$$= \quad -ca^2\int_0^{2\pi}\cos^2\gamma d\gamma\Big(\frac{2}{5}(-(-\frac{1}{3})) + \frac{2}{5}\Big[-\frac{\cos^3\beta}{3}\Big]_0^{\frac{\pi}{2}}\Big)$$

$$= \quad -ca^2\frac{2}{15}\int_0^{2\pi}\cos^2\gamma d\gamma = -ca^2\frac{2}{15}\Big[\frac{\gamma}{2} + \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{2\pi}{15}ca^2$$

$-cB_1 : b * (-2bc)$:

$$-2b^2c\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta\sin^3\beta\cos^2\beta\sin^2\gamma = -\frac{4}{15}b^2c\Big[\frac{\gamma}{2} - \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{4}{15}\pi$$

$-cB_2 : -c * b^2$:

$$-cb^2\int_0^{2\pi}\sin^2\gamma d\gamma \int_0^{\frac{\pi}{2}} \sin^3\beta\sin^2\beta d\beta = -cb^2\frac{2}{15}\Big[\frac{\gamma}{2} - \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{2\pi}{15}cb^2$$

$-cC : -cc^2$:

$$-c^3\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos^4\beta\sin\beta d\beta = -c^3\int_0^{2\pi} d\gamma\Big[-\frac{\cos^5\beta}{5}\Big]_0^{\frac{\pi}{2}} = -\frac{2\pi}{5}cc^2$$

The integral of the terms of $2ab$, $-2bc$ and $-2ca$ are all zero respectively. Therefore we get as follows :

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\int_0^{2\pi} g^3\Delta\\ &= (-\frac{2}{15}-\frac{4}{15})\pi ca^2+(-\frac{2}{15}-\frac{4}{15})\pi cb^2-\frac{2\pi}{5}cc^2\\ &= -\frac{2\pi c}{5}\Big(a^2+b^2+c^2\Big)=-\frac{2\pi c}{5},\quad \text{where } a^2+b^2+c^2=1\end{aligned}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi} g^2h\Delta=-\frac{2\pi}{15}\Big(3c^2c'+2aa'c+2bb'c+a^2c'+b^2c'\Big)=-\frac{2\pi}{15}\Big\{2c(cc'+aa'+bb')+c'(c^2+a^2+b^2)\Big\}=-\frac{2\pi}{15}c',$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\int_0^{2\pi} ghl\Delta\\ &= -\frac{2\pi}{15}\Big(3cc'c''+aa'c''+aa''c'+a'a''c+bb'c''+bb''c'+b'b''c\Big)\\ &= -\frac{2\pi}{15}\Big\{c''(cc'+aa'+bb')+c'(a''a+b''b+c''c)+c(a''a'+b''b'+c''c')\Big\}=0.\end{aligned}$$

Then in brief :

$$\iint g^3\Delta=-\frac{2\pi c}{5},\quad \iint g^2h\Delta=-\frac{2\pi c'}{15},\quad \iint ghl\Delta=0.$$

We get as the same as above.

$$\iint h^3\Delta=-\frac{2\pi c'}{5},\quad \iint l^3\Delta=-\frac{2\pi c''}{5},$$

$$\iint gh^2\Delta=-\frac{2\pi c}{15},\quad \iint gl^2\Delta=-\frac{2\pi c}{15},\quad \iint g^2l\Delta=-\frac{2\pi c''}{15},$$

$$\iint h^2l\Delta=-\frac{2\pi c''}{15},\quad \iint hl^2\Delta=-\frac{2\pi c'}{15}.$$

We show in brief :

$$\frac{2\pi}{3}\sum\frac{r^3}{\alpha^5}fr\equiv K,\quad \frac{2\pi}{15}\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\equiv k. \tag{57}$$

These coefficients were replaced later with (75), in Poisson[47], p.46, p.140.)

By using (57), we get from (53) as follows :

$$\begin{cases}P=-K\Big(c+\frac{du}{dx}c+\frac{du}{dy}c'+\frac{du}{dz}c''\Big)-k\Big(3\frac{du}{dx}c+\frac{du}{dy}c'+\frac{du}{dz}c''+\frac{dv}{dx}c'+\frac{dv}{dy}c+\frac{dw}{dx}c''+\frac{dw}{dz}c\Big),\\ Q=-K\Big(c'+\frac{dv}{dx}c+\frac{dv}{dy}c'+\frac{dv}{dz}c''\Big)-k\Big(\frac{dv}{dx}c+3\frac{dv}{dy}c'+\frac{dv}{dz}c''+\frac{du}{dx}c'+\frac{du}{dy}c+\frac{dw}{dy}c''+\frac{dw}{dz}c'\Big),\\ R=-K\Big(c''+\frac{dw}{dx}c+\frac{dw}{dy}c'+\frac{dw}{dz}c''\Big)-k\Big(\frac{dw}{dx}c+\frac{dw}{dy}c'+3\frac{dw}{dz}c''+\frac{du}{dx}c''+\frac{du}{dz}c+\frac{dv}{dy}c''+\frac{dv}{dz}c'\Big).\end{cases} \tag{58}$$

- § 8.

$$(3)_{Pe}\begin{cases}X_\rho=\frac{dP_1}{dz}+\frac{dP_2}{dy}+\frac{dP_3}{dx},\\ Y_\rho=\frac{dQ_1}{dz}+\frac{dQ_2}{dy}+\frac{dQ_3}{dx},\\ Z_\rho=\frac{dR_1}{dz}+\frac{dR_2}{dy}+\frac{dR_3}{dx}.\end{cases}$$

- § 10.

$$(4)_{Pe}\begin{cases}X_1+P_1c''+P_2c'+P_3c=0,\\ Y_1+Q_1c''+Q_2c'+Q_3c=0,\\ Z_1+R_1c''+R_2c'+R_3c=0.\end{cases}$$

where c, c' and c'' are cosines of the angles formed between the original coordinates x, y, z and the normal line to the surface of separation.

- § 14.

Les équations (3) et (4) conviennent aussi à l'état primitif du corps ; et pour les appliquer à ce cas particulier, il suffit d'y faire $u=0,\ v=0,\ w=0$, et d'y supprimer toutes les forces données, extérieures ou intérieures. On a alors

$$R_1 = Q_2 = P_3 = -K;$$

les six autres quantités P_1, Q_1, etc., sont nulles, et les six équations (3) et (4) se réduisent à quatre, savoir :

$$\frac{dK}{dx}=0, \frac{dK}{dy}=0, \frac{dK}{dz}=0, K=0.$$

D'après les trois premières, la quantité K est une constante qui est nulle en vertu de la dernière. En remettant donc pour K ce que cette lettre représente (no.6)[12], et supprimant le facteur constant $\frac{2\pi}{3\alpha^5}$, on aura

$$\sum r^3 fr = 0$$

Ainsi, dans l'état du corps qu'on peut regarder comme son état naturel, où il n'est soumis qu'à l'action mutuelle de ses molécules, due à leur attraction et à la chaleur, les intervalles qui les séparent doivent être tels que cette équation ait lieu pour tous les points du corps. Si l'on y introduit une nouvelle quantité de chaleur, ce qui augumentera, pour la méme distance, l'intensité de la force répulsive, sans changer celle de la force attractive, il faudra que les intervalles moléculaires augmentent de manière que cette équation continue de subsister; et de lá vient la dilation calorifique, différente dans les différentes matières, à cause que la fonction fr n'y est pas la mème.

Cette équation donne lieu de faire une remarque importante ; c'est que les sommes $\sum$ du no.6, que représentent *les lettres K et k, ne peuvent être changées en des intégrales,* quoique la variable r croisse dans chacune d'elles par de très-petites différences égales à α ; car si cette transformation était possible, k serait zéro en même temps que K ; d'où il résulterait qu'après le changement de forme du corps, les force $P,\ Q,\ R$, seraient nulles comme auparavant, et que des forces données qui agiraient sur le corps ne pourraient se faire équilibre, ce qui est inadmissible. Pour faire voir que k s'évanouirait au même temps que K, observons qu'on aurait

$$K = \frac{2\pi}{3}\int_0^\infty \frac{r^3}{\alpha^6} fr dr, \quad k = \frac{2\pi}{15}\int_0^\infty \frac{r^5}{\alpha^6} d.\frac{1}{r} fr, \tag{59}$$

en multipliant sous les signes $\sum$ par $\frac{dr}{\alpha}$, et remplaçant ces signes par ceux de l'intégration. Or, si l'on intègre par partie, et si l'on fait attention que fr est nulle aux deux limites, il en résultera

$$k = -\frac{2\pi}{3}\int_0^\infty \frac{r^3}{\alpha^6} fr dr = -K \tag{60}$$

ce qui montre que la quantité K étant nulle, on aurait aussi $k=0$. [45, p.398-399, § 14]

- § 16.

Je substitute, en outre, dans les équations $(3)_{Pe}$ à la place de P, Q, etc., leurs valeurs, et je suppose le corps homogène; en observant que $K=0$, il vient

$$(6)_{Pe} \quad \begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = 0, \end{cases} \tag{61}$$

[12] § 6.

a^2 étant un coefficient, égal à $\frac{3k}{\rho}$. Ces équations ont la même forme que celles qui ont été données par M.Navier[13], et qu'il a obtenues en partant de l'hypothèse que les molécules du corps, après son changement de forme, s'attirent proportionnellement aux accroissements des leurs distances mutuelles; et en admettant, de plus, que les résultantes de ces forces peuvent s'exprimer *par des intégrales, ce qui rendrait nul le coefficient* a^2, ainsi qu'on l'a vu plus haut. Les équations relatives à la surface, formées de la même manière, se trouvent aussi dans le Mémoire de M.Navier. [47, p.403-4,§16]

We can see that $(6)_{P^e}$ (= (61)) is able to be modified to (21) as follows :

$$\begin{cases} X - \frac{d^2u}{dt^2} + \frac{a^2}{3}\left(3\frac{d^2u}{dx^2} + 2\frac{d^2v}{dydx} + 2\frac{d^2w}{dzdx} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + \frac{a^2}{3}\left(3\frac{d^2v}{dy^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dzdy} + \frac{d^2v}{dx^2} + \frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + \frac{a^2}{3}\left(3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) = 0, \end{cases} \tag{62}$$

3.2.3. General principle and equations in elastic solid and fluid.

Poisson proposed two constants k and K in his compressible fluid equations in 1829 and was issured in 1831([47, p.46, p.140]) ,

$$(3\text{-}8)_{Pf} \qquad k \equiv \frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} = \frac{2\pi}{15}\sum \frac{1}{4\pi\varepsilon^3} r^3 \frac{d.\frac{1}{r}fr}{dr}, \quad K \equiv \frac{1}{6\varepsilon^3}\sum rfr = \frac{2\pi}{3}\sum \frac{rfr}{4\pi\varepsilon^3}, \tag{63}$$

ε : *la grandeur moyenne des intervalles moléculaires autour du point* M. (the mean value of the molecular intervals around the point M.)([47], p.141).

We summarize Poisson's deduction of k and K in [47], which is a little different with [45, p.368-405, § 1-§ 16].[14]

- § 15. We put ω : the area of the holizontal section ; along this section, an vertical cylindar elevates, the height of which equals at least to the radius of the molecular activity. We call B : the cylindar : the molecular action of A' on it of B, divided with ω, is the pressure by A' on A, related to the unit of the surface and relative to the point M.

  We it with $N\omega$ such that N the pressure related to the unit of the surface ; and because the vertical component of the force : fr acting at the point m and passed from below to upper, is $\frac{z}{r}fr$, we take :

$$N\omega = \sum \frac{z}{r} fr$$

  where the sum $\sum$ covers all the points m of B and m' of A'.
- § 16. We put N : the pressure, ε : the mean value of the molecular intervals around the point M as above. We put ν : a propotional number to the volume : ωz.

$$N\omega = \sum \frac{\nu z}{r} fr$$

$$\nu = \frac{\omega z}{\varepsilon^3}$$

$$(3\text{-}1)_{Pf} \quad N = \frac{1}{\varepsilon^3}\sum \frac{z^2}{r} fr$$

If we call μ the mass of a molecule, or its mean value, the mass of the cylindar : ωz turns equal to $\nu\mu$, and the ratio : $\frac{\nu\mu}{\omega z}$ expresses the density. Hence, we put it with ρ, and put its value for ν, we have :

$$\rho = \frac{\mu}{\varepsilon^3}$$

- § 17. We see also that the quantity : $\frac{z^2}{r}fr$ obeys under the sign $\sum$ being null for all the points of the plane moved by M, the sum which it makes, become $\frac{1}{2}$ of the same sum extended to all the points of A and of A'. Moreover, r^2, which is the square of the distance from M' to the three planes of the rectangle passing through M, and the sum $\sum \frac{z^2}{r}fr$ having the same sum which we

[13] By Poisson's footnote : Tome VII de ces Mémoires, which is Navier[26].

[14] In Poisson [47], the title of the chaper 3 is "*Calcul des Pressions dans les Corps élastiques ; équations defferentielles de l'équiblibre et du mouvement de ces Corps.*"

replace successively z^2 with the two another squares : x^2 and y^2, then it turns that it equals to $\frac{1}{3}\sum\frac{r^2}{r}fr$. After these considerations, we have as follows :

$$(3\text{-}2)_{Pf} \quad N=\frac{1}{\varepsilon^3}\sum\frac{z^2}{r}fr=\frac{1}{2}\times\frac{1}{3}\times\frac{1}{\varepsilon^3}\sum\frac{r^2}{r}fr=\frac{1}{6\varepsilon^3}\sum rfr; \tag{64}$$

- § 20. (This section corresponds to that from § 2 to § 4 in [45] describing the elastic solid.)

$$(3\text{-}6)_{Pf} \quad \begin{cases} x'=ax_1+by_1-cz_1,\\ y'=a'x_1+b'y_1-c'z_1,\\ z'=a''x_1+b''y_1-c''z_1 \end{cases}$$

In sum,

$$\begin{bmatrix} x'\\ y'\\ z' \end{bmatrix}=\begin{bmatrix} a & b & -c\\ a' & b' & -c'\\ a'' & b'' & -c'' \end{bmatrix}\begin{bmatrix} x_1\\ y_1\\ z_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1\\ y_1\\ z_1 \end{bmatrix}=\begin{bmatrix} a & a' & a''\\ b & b' & b''\\ -c & -c' & -c'' \end{bmatrix}\begin{bmatrix} x'\\ y'\\ z' \end{bmatrix}, \tag{65}$$

where, 9 coefficients $a,\ b,\ \cdots$ are the cosines of the angles which $x_1,\ y_1$ and the extension of the axis z_1, with the axis of $x,\ y,\ z$, and these cosines are given.

$$r_1^2=(\varphi+\varphi')^2+(\psi+\psi')^2+(\theta+\theta')^2$$

where for abbreviation :

$$\begin{cases} ax_1+by_1-cz_1\equiv\varphi,\\ a'x_1+b'y_1-c'z_1\equiv\psi,\\ a''x_1+b''y_1-c''z_1\equiv\theta, \end{cases} \qquad \begin{cases} \varphi\frac{du}{dx}+\psi\frac{du}{dy}+\theta\frac{du}{dz}\equiv\varphi',\\ \varphi\frac{dv}{dx}+\psi\frac{dv}{dy}+\theta\frac{dv}{dz}\equiv\psi',\\ \varphi\frac{dw}{dx}+\psi\frac{dw}{dy}+\theta\frac{dw}{dz}\equiv\theta' \end{cases} \tag{66}$$

In sum,

$$\begin{bmatrix} \varphi\\ \psi\\ \theta \end{bmatrix}\equiv\begin{bmatrix} a & b & -c\\ a' & b' & -c'\\ a'' & b'' & -c'' \end{bmatrix}\begin{bmatrix} x_1\\ y_1\\ z_1 \end{bmatrix}, \quad \begin{bmatrix} \varphi'\\ \psi'\\ \theta' \end{bmatrix}\equiv\begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz}\\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz}\\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix}\begin{bmatrix} \varphi\\ \psi\\ \theta \end{bmatrix}=\nabla\mathbf{u}\cdot\begin{bmatrix} \varphi\\ \psi\\ \theta \end{bmatrix}$$

$$\omega P=-\sum\frac{\varphi+\varphi'}{r_1}fr_1,\quad \omega Q=-\sum\frac{\psi+\psi'}{r_1}fr_1\quad \omega R=-\sum\frac{\theta+\theta'}{r_1}fr_1.$$

for the components of the total action of A' on B, in covering the summation $\sum$ to the all points m' of A' and to the all points m of B. Because the function fr_1 is regarded as positive or negative, in accordance with the distance : r_1, the force which it represents, become repulsive or atractive, the components act in the direction of x, y and z, positive or negative, with their values above turn into positive or negative.

$$P=-\frac{1}{\varepsilon^3}\sum\frac{(\varphi+\varphi')z_1}{r_1}fr_1,\quad Q=-\frac{1}{\varepsilon^3}\sum\frac{(\psi+\psi')z_1}{r_1}fr_1\quad R=-\frac{1}{\varepsilon^3}\sum\frac{(\theta+\theta')z_1}{r_1}fr_1.$$

By observing

$$r^2=\varphi^2+\psi^2+\theta^2,$$

and neglecting the quantities of the second order with respect to φ', ψ', θ' then we get :

$$r_1=r+\frac{1}{r}(\varphi\varphi'+\psi\psi'+\theta\theta')$$

At the same degree of approximation, we get as follows :[15]

$$\frac{1}{r_1}fr_1=\frac{1}{r}fr+(\varphi\varphi'+\psi\psi'+\theta\theta')\frac{d.\frac{1}{r}fr}{rdr} \tag{67}$$

$$(3\text{-}7)_{Pf} \quad \begin{cases} P=-\frac{1}{\varepsilon^3}\sum\frac{(\varphi+\varphi')z_1}{r}fr-\frac{1}{\varepsilon^3}\sum(\varphi\varphi'+\psi\psi'+\theta\theta')\varphi z_1\frac{d.\frac{1}{r}fr}{rdr},\\ Q=-\frac{1}{\varepsilon^3}\sum\frac{(\psi+\psi')z_1}{r}fr-\frac{1}{\varepsilon^3}\sum(\varphi\varphi'+\psi\psi'+\theta\theta')\psi z_1\frac{d.\frac{1}{r}fr}{rdr},\\ R=-\frac{1}{\varepsilon^3}\sum\frac{(\omega+\omega')z_1}{r}fr-\frac{1}{\varepsilon^3}\sum(\varphi\varphi'+\psi\psi'+\theta\theta')\omega z_1\frac{d.\frac{1}{r}fr}{rdr}, \end{cases} \tag{68}$$

[15] This equation (67) must equal to elastic solid case (52) in Poisson [45], in which we correct his misprint.

TABLE 8. The 63 coefficients of the components of $-H$

	$x_1^2 z_1^2$ of E	$y_1^2 z_1^2$ of F	$z_1^2 z_1^2 (= z_1^4)$ of G	number of term
$\frac{du}{dx}$	$a(ca+2ca) = ca^2 + 2caa = 3ca^2$	$b(cb+2cb) = cb^2 + 2cbb = 3cb^2$	$cc^2 = c^3$	7
$\frac{dv}{dy}$	$a'(ca'+2c'a) = ca'^2 + 2c'aa'$	$b'(cb'+2c'b) = cb'^2 + 2c'bb'$	cc'^2	7
$\frac{dw}{dz}$	$a''(ca''+2c''a) = ca''^2 + 2c''aa''$	$b''(cb''+2c''b) = cb''^2 + 2c''bb''$	cc''^2	7
$\frac{du}{dy} + \frac{dv}{dx}$	$a(c'a+2ca') = c'a^2 + 2caa'$	$b(c'b+2cb') = c'b^2 + 2cbb'$	$cc'c = c^2c'$	$7 \times 2 = 14$
$\frac{du}{dz} + \frac{dw}{dx}$	$a(c''a+2ca'') = c''a^2 + 2caa''$	$b(c''b+2cb'') = c''b^2 + 2cbb''$	$ccc'' = c^2c''$	$7 \times 2 = 14$
$\frac{dv}{dz} + \frac{dw}{dy}$	$ca'a'' + c'aa'' + c''aa'$	$cb'b'' + cbb'' + c''bb'$	$cc'c''$	$7 \times 2 = 14$
number of term	27	27	9	63

$$\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = -\frac{1}{\varepsilon^3}\sum\left(\begin{bmatrix} (\varphi+\varphi')z_1 & (\varphi\varphi'+\psi\psi'+\theta\theta')\varphi z_1 \\ (\psi+\psi')z_1 & (\varphi\varphi'+\psi\psi'+\theta\theta')\psi z_1 \\ (\omega+\omega')z_1 & (\varphi\varphi'+\psi\psi'+\theta\theta')\omega z_1 \end{bmatrix} \begin{bmatrix} \frac{fr}{r} \\ \frac{d.\frac{1}{r}fr}{rdr} \end{bmatrix}\right)$$

3.2.4. **K and first summation of P, Q, R in elastic solid.**

- § 21. (This section corresponds to that from § 5 to § 7 in [45] describing the elastic solid.)

$$\sum \frac{z^2}{r} fr = \frac{1}{6}\sum rfr,$$

$$\begin{cases} \sum \frac{(\varphi+\varphi')z_1}{r} fr = -\frac{1}{6}\left(c + c\frac{du}{dx} + c'\frac{du}{dy} + c''\frac{du}{dz}\right)\sum rfr = -\frac{1}{6}\left(c\left(1+\frac{du}{dx}\right) + c'\frac{du}{dy} + c''\frac{du}{dz}\right)\sum rfr, \\ \sum \frac{(\psi+\psi')z_1}{r} fr = -\frac{1}{6}\left(c' + c\frac{dv}{dx} + c'\frac{dv}{dy} + c''\frac{dv}{dz}\right)\sum rfr = -\frac{1}{6}\left(c\frac{dv}{dx} + c'\left(1+\frac{dv}{dy}\right) + c''\frac{dv}{dz}\right)\sum rfr, \\ \sum \frac{(\omega+\omega')z_1}{r} fr = -\frac{1}{6}\left(c'' + c\frac{dw}{dx} + c'\frac{dw}{dy} + c''\frac{dw}{dz}\right)\sum rfr = -\frac{1}{6}\left(c\frac{dw}{dx} + c'\frac{dw}{dy} + c''\left(1+\frac{dw}{dz}\right)\right)\sum rfr, \end{cases}$$

3.2.5. **k and second summation of P, Q, R in elastic solid.**

We call H the second summation in P of (68)(= (3-7)$_{Pf}$) such as :

$$H \equiv \sum (\varphi\varphi' + \psi\psi' + \theta\theta')\varphi z_1 \frac{d.\frac{1}{r}fr}{rdr}$$

$$\sum x_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} \equiv E, \quad \sum y_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} \equiv F, \quad \sum z_1^4 \frac{d.\frac{1}{r}fr}{rdr} \equiv G,$$

We get the 63 coefficients of components of $-H$ as in Table 8. The sums of E, F and G are equal for A and for A', because the terms related to the plane made of x_1 and y_1, become equal to zero by taking the differencial : we can take the volume of the total body, and take successively as $\frac{1}{2}$ of its value. When we regard the body as homogeneous in the volume of the sphare of the molecular activity, we get as follows :

$$\begin{cases} \sum z_1^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum y_1^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum x_1^4 \frac{d.\frac{1}{r}fr}{rdr}, \\ \sum y_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x_1^2 y_1^2 \frac{d.\frac{1}{r}fr}{rdr}. \end{cases}$$

$$\begin{cases} \sum z'^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum y'^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum x'^4 \frac{d.\frac{1}{r}fr}{rdr} = 2G, \\ \sum y'^2 z'^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x'^2 z'^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x'^2 y'^2 \frac{d.\frac{1}{r}fr}{rdr} = 2E = 2F. \end{cases} \quad (69)$$

From (65) we get :

$$\begin{cases} x_1 = ax' + a'y' + a''z' \\ y_1 = bx' + b'y' + b''z' \\ z_1 = -cx' - c'y' - c''z' \end{cases}$$

In Table 9,

TABLE 9. The coefficients by the combination of the terms of $z_1^2 \times z_1^2$ in G

	$c^2x'^2$	$c'^2y'^2$	$c''^2z'^2$	$2cc'x'y'$	$2c'c''y'z'$	$2cc''x'z'$
$c^2x'^2$	$c^4x'^4$	$2c^2c'^2x'^2y'^2$	$2c^2c''^2x'^2z'^2$	$4c^3c'x'^3y'$	$4c^2c'c''x'^2y'z'$	$4c^3c''x'^3z'$
$c'^2y'^2$		$c'^4y'^4$	$2c'^2c''^2y'^2z'^2$	$4cc'^3x'y'^3$	$4c'^3c''y'^3z'$	$4cc'^2c''x'y'^2z'$
$c''^2z'^2$			$c''^4z'^4$	$4cc'c''^2x'y'z'^2$	$4c'c''^3y'z'^3$	$4cc''^3x'z'^3$
$2cc'x'y'$				$4c^2c'^2x'^2y'^2$	$8cc'^2c''x'y'^2z'$	$8c^2c'c''x'^2y'z'$
$2c'c''y'z'$					$4c'^2c''^2y'^2z'^2$	$8cc'c''^2x'y'z'^2$
$2cc''x'z'$						$4c^2c''^2x'^2z'^2$

TABLE 10. The 21 coefficients by the combination of the terms of $z_1^2 \times z_1^2$ in G

	$c^2x'^2$	$c'^2y'^2$	$c''^2z'^2$	$2cc'x'y'$	$2c'c''y'z'$	$2cc''x'z'$
$c^2x'^2$	$c^4x'^4$	$2c^2c'^2x'^2y'^2$	$2c^2c''^2x'^2z'^2$			
$c'^2y'^2$		$c'^4y'^4$	$2c'^2c''^2y'^2z'^2$			
$c''^2z'^2$			$c''^4z'^4$			
$2cc'x'y'$				$4c^2c'^2x'^2y'^2$		
$2c'c''y'z'$					$4c'^2c''^2y'^2z'^2$	
$2cc''x'z'$						$4c^2c''^2x'^2z'^2$

$$
\begin{aligned}
&4(c^3c'x'^3y' + c^2c'c''x'^2y'z' + c^3c''x'^3z' + cc'^3x'y'^3 + c'^3c''y'^3z' + cc'^2c''x'y'^2z' \\
&+cc'c''^2x'y'z'^2 + c'c''^3y'z'^3 + cc''^3x'z'^3 + 2cc'^2c''x'y'^2z' + 2c^2c'c''x'^2y'z' + 2cc'c''^2x'y'z'^2) \\
= 4\Big[&(c^2x'^2 + c'^2y'^2 + c''^2z'^2)(cc'x'y' + c'c''y'z' + cc''x'z') + 2(c^2c'c''x'y'z' + cc'^2c''x'y'^2z' + cc'c''x'y'z'^2)\Big]
\end{aligned}
$$

Hence, we can consider only the elements of Table 10. From (69) and the 21 elements in the upper-trianglar matrix including the diagonal of Table 10 , we get :

$$
\begin{aligned}
G &= \sum z_1^4 \frac{d.\frac{1}{r}fr}{rdr} \\
&= \frac{1}{2}\sum \frac{d.\frac{1}{r}fr}{rdr}\Big[\big(c^4x'^4 + c'^4y'^4 + c''^4z'^4\big) + 6\big(c^2c'^2x'^2y'^2 + c^2c''^2y'^2z'^2 + c'^2c''^2z'^2x'^2\big)\Big] \\
&= \frac{1}{2}\Big[2G\big(c^4 + c'^4 + c''^4\big) + 6 \cdot 2F\big(c^2c'^2 + c^2c''^2 + c'^2c''^2\big)\Big] \\
&= G\big(c^4 + c'^4 + c''^4\big) + 6F\big(c^2c'^2 + c^2c''^2 + c'^2c''^2\big).
\end{aligned} \tag{70}
$$

Here, we put for convenience' sake as follows :

$$
\alpha \equiv c^4 + c'^4 + c''^4, \qquad \beta \equiv c^2c'^2 + c^2c''^2 + c'^2c''^2.
$$

Because of[16]

$$\begin{cases} c^2+c'^2+c''^2=1, \\ c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=1, \end{cases}$$

$$\Rightarrow \quad \begin{cases} (c^2+c'^2+c''^2)^2=1, \\ c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=1 \Rightarrow \alpha = 1-2\beta, \end{cases}$$

then from (70), we get $G=\alpha G+6\beta F$ and then :

$$G=(1-2\beta)G+6\beta F \qquad \Rightarrow \qquad 2\beta G=6\beta F,$$

it turns at last :

$$G=3F.$$

Moreover, because of $r_1^2=x_1^2+y_1^2+z_1^2$, we get

$$\sum r^4 \frac{d.\frac{1}{r}fr}{rdr} \tag{71}$$

$$= \sum x_1^4\frac{d.\frac{1}{r}fr}{rdr}+\sum y_1^4\frac{d.\frac{1}{r}fr}{rdr}+\sum z_1^4\frac{d.\frac{1}{r}fr}{rdr}+2\sum x_1^2y_1^2\frac{d.\frac{1}{r}fr}{rdr}+2\sum x_1^2z_1^2\frac{d.\frac{1}{r}fr}{rdr}+2\sum y_1^2z_1^2\frac{d.\frac{1}{r}fr}{rdr}, \tag{72}$$

then (71) turns from (69) :

$$\frac{1}{2}\sum r^3\frac{d.\frac{1}{r}fr}{dr}=3G+6F=5G=15E=15F.$$

$$G=\frac{1}{10}\sum r^3\frac{d.\frac{1}{r}fr}{dr}, \quad E=F=\frac{1}{30}\sum r^3\frac{d.\frac{1}{r}fr}{dr},$$

As the common factor, we take $\frac{1}{30}$ then :

$$H=-\frac{1}{30}\sum r^3\frac{d.\frac{1}{r}fr}{dr}\Big[c\Big(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)+c'\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)+c''\Big(\frac{du}{dz}+\frac{dw}{dx}\Big)\Big]. \tag{73}$$

The second summation contained in Q of (68) (= (3-7)$_{Pf}$) is deduced from H with the permutation of u and v, x and y, c and c', and samely in R with the permutation of u and w, x and z, c and c''. By these manner, the equations (68) (= (3-7)$_{Pf}$) turn out as follows :

$$\begin{cases} P=\Big[K(1+\frac{du}{dx})+k\Big(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big]c+\Big[K\frac{du}{dy}+k\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\Big]c'+\Big[K\frac{du}{dz}+k\Big(\frac{du}{dz}+\frac{dw}{dx}\Big)\Big]c'', \\ Q=\Big[K(1+\frac{dv}{dy})+k\Big(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big]c'+\Big[K\frac{dv}{dx}+k\Big(\frac{dv}{dx}+\frac{du}{dy}\Big)\Big]c+\Big[K\frac{dv}{dz}+k\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\Big]c'', \\ R=\Big[K(1+\frac{dw}{dz})+k\Big(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\Big)\Big]c''+\Big[K\frac{dw}{dy}+k\Big(\frac{dw}{dy}+\frac{dv}{dz}\Big)\Big]c'+\Big[K\frac{dw}{dx}+k\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\Big]c, \end{cases} \tag{74}$$

where, for abbreviation, he uses :

$$(3\text{-}8)_{Pf} \qquad k\equiv\frac{1}{30\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr}=\frac{2\pi}{15}\sum\frac{1}{4\pi\varepsilon^3}r^3\frac{d.\frac{1}{r}fr}{dr}, \quad K\equiv\frac{1}{6\varepsilon^3}\sum rfr=\frac{2\pi}{3}\sum\frac{rfr}{4\pi\varepsilon^3}, \tag{75}$$

[16]We corrected Poisson's mistake :

$$c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=0 \quad \Rightarrow \quad =1.$$

Because if

$$c^2+c'^2+c''^2=1$$

then we get clealy

$$c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=1$$

Inversely, if the equation equals to 0, then

$$G=-2\beta G+6\beta F \quad \Rightarrow \quad (1+2\beta)G=6\beta F \quad \Rightarrow \quad G=\frac{6\beta}{1+2\beta}F.$$

Then we can't get $G=3F$.

- § 22. We get the general equations as follows :

$$(3\text{-}9)_{Pf}\quad \begin{cases} P = P_1c'' + P_2c' + P_3c, \\ Q = Q_1c'' + Q_2c' + Q_3c, \\ R = R_1c'' + R_2c' + R_3c \end{cases} \Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} c'' \\ c' \\ c \end{bmatrix},$$

then we get the tensor from (74) as follows :

$$\begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} K\frac{du}{dz} + k\left(\frac{du}{dz} + \frac{dw}{dx}\right) & K\frac{du}{dy} + k\left(\frac{du}{dy} + \frac{dv}{dx}\right) & K(1+\frac{du}{dx}) + k\left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) \\ K\frac{dv}{dz} + k\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & K(1+\frac{dv}{dy}) + k\left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right) & K\frac{dv}{dx} + k\left(\frac{dv}{dx} + \frac{du}{dy}\right) \\ K(1+\frac{dw}{dz}) + k\left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right) & K\frac{dw}{dy} + k\left(\frac{dw}{dy} + \frac{dv}{dz}\right) & K\frac{dw}{dx} + k\left(\frac{dw}{dx} + \frac{du}{dz}\right) \end{bmatrix}$$

- § 23.
  When we suppose that the initial state of the elastic solid is natural, it turns $K = 0$, so we get as follows :

$$(3\text{-}11)_{Pf}\quad \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} k\left(\frac{du}{dz} + \frac{dw}{dx}\right) & k\left(\frac{du}{dy} + \frac{dv}{dx}\right) & k\left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) \\ k\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & k\left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right) & k\left(\frac{dv}{dx} + \frac{du}{dy}\right) \\ k\left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right) & k\left(\frac{dw}{dy} + \frac{dv}{dz}\right) & k\left(\frac{dw}{dx} + \frac{du}{dz}\right) \end{bmatrix}$$

- § 24.

$$(3\text{-}12)_{Pf}\quad \begin{cases} X\rho = \frac{dP_1}{dz} + \frac{dP_2}{dy} + \frac{dP_3}{dx}, \\ Y\rho = \frac{dQ_1}{dz} + \frac{dQ_2}{dy} + \frac{dQ_3}{dx}, \\ Z\rho = \frac{dR_1}{dz} + \frac{dR_2}{dy} + \frac{dR_3}{dx}, \end{cases} \Rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{\rho} \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dz} \\ \frac{d}{dy} \\ \frac{d}{dx} \end{bmatrix} \qquad (76)$$

- § 27.
  In homogeneous case, δ means the difference of the contraction or dilation :

$$\frac{r' - r}{r} \equiv -\delta$$

$$P = -5k\delta c, \quad Q = -5k\delta c', \quad R = -5k\delta c'';$$

$$K = -5k\delta.$$

Replacing ε and r of K in (75) ($= (3\text{-}8)_{Pf}$) with ε' and r',

$$K = \frac{1}{6\varepsilon'^3}\sum r'fr'$$

and r' and ε' with

$$r' = r - r\delta, \quad \varepsilon' = z - \varepsilon\delta.$$

[17] For δ is very small value, we can develope K by the convergent series, ordered by following power of δ, and neglecting the bigger power than the first, then it turns out as follows :

$$K = \frac{1+\delta}{6\varepsilon^3}\sum rfr - \frac{\delta}{6\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr},$$

then, because $\sum rfr = 0$, by the condition of natural state,

$$K = -\frac{\delta}{6\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr} = -5k\delta.$$

- § 31. Finally, he assumes isotropic elasticity in natural state and the perpendicular pressure on the surface of corps.

---

[17]Then we get :

$$K = \frac{1}{6(z - \varepsilon\delta)^3}\sum r(1-\delta)f(r - r\delta)$$

Je substitute, en outre, dans les équations (12)[18], à la place de $P_1,\ Q_1,\cdots$, leurs valeurs. Je supose le corps homogène, et je prends alors pour son état naturel auquel répondent les coordonées $x,\ y,\ z$, du point quelconque M, un état dans lequel la surface du corps est soumise à une pression normale et la même en tous ses points. En la représentant par Π, on aura $K=\Pi$ (§ 27). La quantite k étant négative (même § 27) et indépendante de $x,\ y,\ z$, je fais, pour abréger

He puts Π the normal pressure on the corps, and for abbreviation, he uses :

$$-\frac{5k}{\rho}\equiv a^2$$

then the motional equations of elastic corps are as follows :

$$\begin{cases} X-\frac{d^2u}{dt^2}+a^2\left(\frac{d^2u}{dx^2}+\frac{2}{3}\frac{d^2v}{dydx}+\frac{2}{3}\frac{d^2w}{dzdx}+\frac{1}{3}\frac{d^2u}{dy^2}+\frac{1}{3}\frac{d^2u}{dz^2}\right)=\frac{\Pi}{\rho}\frac{d^2u}{dx^2},\\ Y-\frac{d^2v}{dt^2}+a^2\left(\frac{d^2v}{dy^2}+\frac{2}{3}\frac{d^2u}{dxdy}+\frac{2}{3}\frac{d^2w}{dzdy}+\frac{1}{3}\frac{d^2v}{dx^2}+\frac{1}{3}\frac{d^2v}{dz^2}\right)=\frac{\Pi}{\rho}\frac{d^2v}{dy^2},\\ Z-\frac{d^2w}{dt^2}+a^2\left(\frac{d^2w}{dz^2}+\frac{2}{3}\frac{d^2u}{dxdz}+\frac{2}{3}\frac{d^2v}{dydz}+\frac{1}{3}\frac{d^2w}{dx^2}+\frac{1}{3}\frac{d^2w}{dy^2}\right)=\frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases} \tag{77}$$

3.2.6. **Fluid pressure in motion.**

- § 63.

[19] Poisson's tensor of the pressures in fluid reads as follows :

$(7\text{-}7)_{Pf}$

$$\begin{bmatrix} U_1 & U_2 & U_3\\ V_1 & V_2 & V_3\\ W_1 & W_2 & W_3\end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{du}{dx}\\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right)\\ p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right)\end{bmatrix},$$

$$(k+K)\alpha=\beta,\quad (k-K)\alpha=\beta',\quad p=\psi t=K,\quad \text{then}\quad \beta+\beta'=2k\alpha, \tag{78}$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here we can replace the first column with the third one, then we see easily the conventional style of array as follows :

$$\begin{bmatrix} U_3 & U_2 & U_1\\ V_3 & V_2 & V_1\\ W_3 & W_2 & W_1\end{bmatrix} = \begin{bmatrix} p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{du}{dx} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right)\\ \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dv}{dy} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\\ \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dw}{dz}\end{bmatrix},$$

The elements of velocity $\mathbf{u}=(u,v,w)$ are :

$$\frac{dx}{dt}=u,\quad \frac{dy}{dt}=v,\quad \frac{dz}{dt}=w$$

$$\begin{cases} \frac{d^2x}{dt^2}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz},\\ \frac{d^2y}{dt^2}=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz},\\ \frac{d^2z}{dt^2}=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz} \end{cases}$$

$$\varpi\equiv-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \tag{79}$$

[20]

$(7\text{-}9)_{Pf}$

$$\begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}),\\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}),\\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}). \end{cases} \tag{80}$$

[18] §24, $(3\text{-}12)_{Pf}$ (= (76)).

[19] In Poisson [47], the title of the chaper 7 is "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

[20] $(7\text{-}9)_{Pf}$ means the equation number with chapter of Poisson [47]

If we put $\mathbf{f} = (X, Y, Z)$ then (80) becomes as follows :

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\beta}{\rho}\Delta \mathbf{u} + \frac{1}{\rho}\nabla \varpi = \mathbf{f} \tag{81}$$

3.2.7. **Stokes' comment on Poisson's fluid equations.**

Stokes comments on Poisson's $(7\text{-}9)_{Pf}$ as follows :

> On this supposition we shall get the value of $\frac{d\Psi t}{dt}$ from that of $R_1' - K$ in the equations of page 140 by putting
>
> $$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = -\frac{1}{3\chi t}\frac{d\chi t}{dt}.$$
>
> We have therefore
>
> $$\alpha\frac{d\chi t}{dt} = \frac{\alpha}{3}(K - 5k)\frac{d\chi t}{\chi t dt}.$$
>
> Putting now for $\beta + \beta'$ its value $2\alpha k$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (82)[21], the expression for ϖ, page 152,[22] becomes
>
> $$\varpi = p + \frac{\alpha}{3}(K + k)(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}).$$
>
> Observing that $\alpha(K + k) = \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$[=(80)] to the equation $(12)_S$[=(97)] of this paper. ([55, p.119])

Namely, by using $\alpha(K + k) = \beta$ in (78) and the followings :

$$\begin{cases} \frac{d\varpi}{dx} = \frac{dp}{dx} + \frac{\alpha}{3}(K + k)\frac{d}{dx}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \\ \frac{d\varpi}{dy} = \frac{dp}{dy} + \frac{\alpha}{3}(K + k)\frac{d}{dy}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \\ \frac{d\varpi}{dz} = \frac{dp}{dz} + \frac{\alpha}{3}(K + k)\frac{d}{dz}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \end{cases}$$

then $(80)(= (7\text{-}9)_{Pf}$) turns out :

$$(7\text{-}9)_{Pf} \quad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \\ \text{where } \varpi = p + \frac{\alpha}{3}(K + k)(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \end{cases}$$

$$\Rightarrow \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K + k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{1}{3}\alpha(K + k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K + k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{1}{3}\alpha(K + k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K + k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{1}{3}\alpha(K + k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$$

$$\Rightarrow (12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$$

Therefore, Poisson contains both compressible and incompressible fluid.

3.3. **Cauchy's deduction of tensor.**

[21] Poisson[47, p.141],

$$(7\text{-}2)_{Pf} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{1}{\chi t}\frac{d\chi t}{dt}. \tag{82}$$

[22] cf. (79)

TABLE 11. Assumptions and model of the system of particles by Cauchy

no	item	ref. equations
1	Mouvement par des forces d'attraction ou de repulsion mutuelle.	
2	La lettre S indiquant une somme de termes semblables, mais relatifs aux diverses molécules $m, m', \cdots$, et la signe $\pm$ devant être réduit au signe + ou au signe − suivant que la masse m sera attirée ou repoussée par molécule m.	$(3)_C$
3-1	L'etat du système de points matériels soit changé.	$(4)_C - (6)_C$
3-2	Les molécules m, m, $m', \cdots$ se déplacent dans l'espace, mais de manière que la distance de deux molécules m et m varie dans un rapport peu différent de l'unité.	$(7)_C$
4	$\xi,\ \eta,\ \zeta$: des fonctions de $a,\ b,\ c$, qui représentent las déplacements très petits et parallèles aux axes d'une molécule quelconque m.	$(8)_C - (11)_C$
5	Les déplacements ξ, η, ζ sont très petits, alors, en considérant ces déplacements comme infiniment petits du premier ordre, et négligeant les infiniment petits du second ordre.	$(12)_C - (31)_C$
6	Les équations qui expriment l'equilibre ou le mouvement du système des masses m, m, $m', \cdots$ soumises, non seulement à leurs attractions ou répulsions mutuelles, mais à de nouvelles forces accélératrice.	$(32)_C - (34)_C$
7	Les sommes comprises dans les formules $(26)_C$ et $(30)_C$ s'evanouissent. Les masses m, m', $m'', \cdots$ étant deux à deux égales entre elles, sont distribuées, symétriquement de part et d'autre de la molécule m.	$(35)_C - (36)_C$
8	Parmi lés sommes comprises dans les formules $(26)_C$, $(30)_C$ et $(31)_C$, toutes celles qui renferment des puissances impaires de $\cos\alpha$, de $\cos\beta$, ou de $\cos\gamma$ s'evanouissent.	$(37)_C - (40)_C$
8-1	Les molécules m, m', $m'', \cdots$ sont distribuées symétriquement par rapport à chacun des trois plans.	
8-2	Deux molécules symétriquement placées à l'egard d'un des trois premiers plans offrent toujours des masses égales.	
9	Les molécules $m, m', m'', \cdots$ primitivement distribuées de la même manière par rapport aux trois plans menés par la molécule m parallèlement aux plans coordonnés.	$(41)_C - (42)_C$
10	Les molécules $m, m', m'', \cdots$ primitivement distribuées autour de la molécule m, de manière que les sommes comprises dans les équations (37), (38), (39) deviennent indépendantes des directions assignées aux axes des x, y, z.	$(43)_C - (48)_C$
11	Les sommes des masses comprises les volemes élémentaires v, v', v'', $\cdots$ soient propotionelles à ces même volumes, et représentées en conséquence par les produits Δv, $\Delta v'$, $\Delta v''$, $\cdots$.	$(49)_C$

3.3.1. Deduction of the equations of accelerate force.

[23] We show Cauchy's 11 assumptions in Table 11, which are numbered from (#1) to (#11) in the followings.

• ¶ 1. At first, we consider that the great number of molecules or material points are arbitrarily distributed in a certain potion of the space and its motion are brought about by the forces of mutual attraction or repulsion. Strict speaking, we must cite Cauchy's assumptions as follows :

(#1). The definition of the various terms are :

- m (in roman style) : mass of this molecule ;
- m, m', m'' (in italic style) : masses of another molecules, which are assumed at a certain time ;
- a, b, c : the coordinate values of the molecule m on the rectanglar coordinates : x, y, z ;
- $a+\Delta a$, $b+\Delta b$, $c+\Delta c$: the coordinate values of the molecule m ;
- r : the distance between m and m (with scalar value) ;
- α, β, γ : the angles formed by the vector of ray : $\mathbf{r}$ with each half axis of the positive coordinates.

• ¶ 2. Cauchy's hypothesis of molecular activities are as follows :

(#2). La lettre **S** indiquant une somme de termes semblables, mais relatifs aux diverses molécules $m, m', \cdots$, et la signe $\pm$ devant être réduit au signe + ou au signe − suivant que la masse m sera attirée ou repoussée par molécule m. Ajoutons que les quantités Δa, Δb, Δc pourront être exprimées en fonction de r et des angles α, β, γ par les formules : [6, p.228]

$$(3)_C \quad \Delta a = r\cos\alpha, \quad \Delta b = r\cos\beta, \quad \Delta c = r\cos\gamma.$$

[23] For convenience' sake, we put "• ¶ (number)" as the paragraph number which is not in his text, but we count and show it, and moreover, we suppose sections.

• ¶ 3.

(#3). Supposons maintenant

- que l'etat du système de points matériels soit changé, et
- que les molécules m, m, m', $\cdots$ se déplacent dans l'espace, mais de manière que la distance de deux molécules m et m varie dans un rapport peu différent de l'unité.

(#4). Soient ξ, η, ζ : des fonctions de a, b, c, qui représentent les déplacements très petits et parallèles aux axes d'une molécule quelconque m ;

- x, y, z ; $x+\Delta x$, $y+\Delta y$, $z+\Delta z$: les coordonnées des molécules m, m dans le nouvel état du système ;
- $r(1+\varepsilon)$: la distance des molécules m, m dans ce nouvel état ;
- ε : la dilation très petite de la longueur r dans le passage du premier état au second ; et l'on aura évidement

$$(4)_C \quad x = a+\xi, \quad y = b+\eta, \quad z = c+\zeta.$$

$$(5)_C \quad \begin{cases} \Delta x = \Delta a + \Delta\xi = r\cos\alpha + \Delta\xi, \\ \Delta y = \Delta b + \Delta\eta = r\cos\beta + \Delta\eta, \\ \Delta z = \Delta c + \Delta\zeta = r\cos\gamma + \Delta\zeta. \end{cases}$$

[24]

$$\begin{aligned}(6)_C \quad r^2(1+\varepsilon)^2 &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) + 2r(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) + (\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2 \\ &= r^2 + 2r(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) + (\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2. \end{aligned}$$

Here we used the following by $(8)_C$:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{(\Delta a)^2}{r^2} + \frac{(\Delta b)^2}{r^2} + \frac{(\Delta c)^2}{r^2} = 1.$$

$$(7)_C \quad 1+\varepsilon = \sqrt{1 + \frac{2}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) + \frac{1}{r^2}((\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2)},$$

We can put the followings with the parallel expressions :

$$(8)_C, (9)_C \quad \cos\alpha = \frac{\Delta a}{r} = \frac{\Delta x}{r(1+\varepsilon)}, \quad \cos\beta = \frac{\Delta b}{r} = \frac{\Delta y}{r(1+\varepsilon)}, \quad \cos\gamma = \frac{\Delta a}{r} = \frac{\Delta z}{r(1+\varepsilon)}.$$

• ¶ 4. After all, the algebraic projections of resultant forces of attractions and repulsions performed by the molecules $m, m', m'', \cdots$ on the molecule m become equal to three products :

$$(10)_C \quad \begin{cases} \mathrm{m}\mathbf{S}\Big\{ \pm m\frac{\Delta x}{r(1+\varepsilon)}\mathrm{f}[r(1+\varepsilon)]\Big\}, \\ \mathrm{m}\mathbf{S}\Big\{ \pm m\frac{\Delta y}{r(1+\varepsilon)}\mathrm{f}[r(1+\varepsilon)]\Big\}, \\ \mathrm{m}\mathbf{S}\Big\{ \pm m\frac{\Delta z}{r(1+\varepsilon)}\mathrm{f}[r(1+\varepsilon)]\Big\}. \end{cases}$$

Here we put the accelerate force as follows :

$$(11)_C \quad \begin{cases} X = \mathbf{S}\Big\{ \pm m\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta x\Big\}, \\ Y = \mathbf{S}\Big\{ \pm m\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta y\Big\}, \\ Z = \mathbf{S}\Big\{ \pm m\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta z\Big\}. \end{cases}$$

les trois products : mX, mY, mZ, et les trois quantités : X, Y, Z représenteront les projections algébriques :

- de la résultante dont il s'agit ;
- de cette résultante divisée par m, ou, qui revient au même, de la force accélératrice qui sollicitera la molécule m et qui sera due aux actions des molécules $m, m', m'', \cdots$

[24]In Cauchy [6, p.228], the second "=" in each statements in $(5)_C$ are putted by "+", and we correct it.

• ¶ 5. The deplacements : $\xi,\ \eta,\ \zeta$ are infinitesimal, then we can neglect these values of second order. (#5). (Omitted.)

$$(7)_C \quad \Rightarrow \quad (12)_C \quad \varepsilon = \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta).$$

$$(13)_C \quad \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)} = (1-\varepsilon)\frac{\mathrm{f}(r)+\varepsilon r\mathrm{f}'(r)}{r} = \frac{\mathrm{f}(r)}{r} + \varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)-\varepsilon\mathrm{f}'(r)}{r} \simeq \frac{\mathrm{f}(r)}{r} + \varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}.$$

Here, we remark the neglection by Cauchy : $(11)_C$ turns into the following from $(5)_C$ and $(13)_C$ as follows :

$$\begin{aligned}
\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta x &= \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}(r\cos\alpha + \Delta\xi) \quad (\because \text{ from } (5)_C\ )\\
&= \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\frac{\mathrm{f}(r)}{r}\Big)\Big(r\cos\alpha + \frac{\Delta\xi r\cos\alpha}{r\cos\alpha}\Big) \quad (\because \text{ from } (13)_C\ )\\
&= \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(1+\frac{\Delta\xi}{r\cos\alpha}\Big)\Big(\frac{\mathrm{f}(r)}{r}\Big)r\cos\alpha\\
&= \Big\{1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\xi}{r\cos\alpha} + \Big(\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\frac{\Delta\xi}{r\cos\alpha}\Big)\Big\}\mathrm{f}(r)\cos\alpha\\
&\simeq \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\xi}{r\cos\alpha}\Big)\mathrm{f}(r)\cos\alpha.
\end{aligned}$$

Samely, we can get the followings :

$$\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta y = \Big\{1+\varepsilon\tfrac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \tfrac{\Delta\eta}{r\cos\beta} + \Big(\varepsilon\tfrac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\tfrac{\Delta\eta}{r\cos\beta}\Big)\Big\}\mathrm{f}(r)\cos\beta \simeq \Big(1+\varepsilon\tfrac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \tfrac{\Delta\eta}{r\cos\beta}\Big)\mathrm{f}(r)\cos\beta,$$

$$\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta z = \Big\{1+\varepsilon\tfrac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \tfrac{\Delta\zeta}{r\cos\gamma} + \Big(\varepsilon\tfrac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\tfrac{\Delta\zeta}{r\cos\gamma}\Big)\Big\}\mathrm{f}(r)\cos\gamma \simeq \Big(1+\varepsilon\tfrac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \tfrac{\Delta\zeta}{r\cos\gamma}\Big)\mathrm{f}(r)\cos\gamma.$$

According to Cauchy's assumption, we get the following $(14)_C$ from $(11)_C$ by combining with $(5)_C$ and $(13)_C$.

$$(14)_C \quad \begin{cases} X = \mathbf{S}\Big\{\pm m\big[1+\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\varepsilon + \frac{\Delta\xi}{r\cos\alpha}\big]\cos\alpha\mathrm{f}(r)\Big\},\\ Y = \mathbf{S}\Big\{\pm m\big[1+\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\varepsilon + \frac{\Delta\eta}{r\cos\beta}\big]\cos\beta\mathrm{f}(r)\Big\},\\ Z = \mathbf{S}\Big\{\pm m\big[1+\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\varepsilon + \frac{\Delta\zeta}{r\cos\gamma}\big]\cos\gamma\mathrm{f}(r)\Big\}. \end{cases}$$

• ¶ 6. From the initial condition, by considering the equilibrium of $X,\ Y,\ Z$, we get some results.

> lorsque le premier état du système des points matériels est état d'équibre, il suffit de remplacer $\xi,\ \eta,\ \zeta$ par zéro dans les formiles $(14)_C$, pour faire évanouir $X,\ Y,\ Z$.

Then we get $(15)_C$ as follows :

$$(15)_C \quad \mathbf{S}[\pm m\cos\alpha\mathrm{f}(r)] = 0, \quad \mathbf{S}[\pm m\cos\beta\mathrm{f}(r)] = 0, \quad \mathbf{S}[\pm m\cos\gamma\mathrm{f}(r)] = 0.$$

$$(16)_C \quad \begin{cases} X = \mathbf{S}\Big\{\pm m\big[\{r\mathrm{f}'(r)-\mathrm{f}(r)\}\varepsilon\cos\alpha + \frac{\mathrm{f}(r)}{r}\Delta\xi\big]\Big\},\\ Y = \mathbf{S}\Big\{\pm m\big[\{r\mathrm{f}'(r)-\mathrm{f}(r)\}\varepsilon\cos\beta + \frac{\mathrm{f}(r)}{r}\Delta\eta\big]\Big\},\\ Z = \mathbf{S}\Big\{\pm m\big[\{r\mathrm{f}'(r)-\mathrm{f}(r)\}\varepsilon\cos\gamma + \frac{\mathrm{f}(r)}{r}\Delta\zeta\big]\Big\}. \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} X\\ Y\\ Z \end{bmatrix} = \mathbf{S}\Big\{\pm m \begin{bmatrix} \varepsilon\cos\alpha & \Delta\xi\\ \varepsilon\cos\beta & \Delta\eta\\ \varepsilon\cos\gamma & \Delta\zeta \end{bmatrix} \begin{bmatrix} (r\mathrm{f}'(r)-\mathrm{f}(r))\varepsilon\\ \frac{\mathrm{f}(r)}{r} \end{bmatrix}\Big\},$$

From $(12)_C$

$$\Rightarrow \quad \begin{bmatrix} X\\ Y\\ Z \end{bmatrix} = \mathbf{S}\Big\{\pm m \begin{bmatrix} \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta)\cos\alpha & \Delta\xi\\ \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta)\cos\beta & \Delta\eta\\ \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta)\cos\gamma & \Delta\zeta \end{bmatrix} \begin{bmatrix} (r\mathrm{f}'(r)-\mathrm{f}(r))\varepsilon\\ \frac{\mathrm{f}(r)}{r} \end{bmatrix}\Big\},$$

$$(16)_C \quad \Rightarrow \quad (17)_C \quad \begin{cases} X = \mathbf{S}\Big\{ \pm m\Big[\Big(\frac{\mathrm{f}(r)}{r} + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\cos^2\alpha\Big)\Delta\xi + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\big(\cos\alpha\cos\beta\Delta\eta + \cos\alpha\cos\gamma\Delta\zeta\big)\Big]\Big\}, \\ Y = \mathbf{S}\Big\{ \pm m\Big[\Big(\frac{\mathrm{f}(r)}{r} + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\cos^2\beta\Big)\Delta\eta + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\big(\cos\beta\cos\gamma\Delta\zeta + \cos\beta\cos\alpha\Delta\xi\big)\Big]\Big\}, \\ Z = \mathbf{S}\Big\{ \pm m\Big[\Big(\frac{\mathrm{f}(r)}{r} + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\cos^2\gamma\Big)\Delta\zeta + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\big(\cos\gamma\cos\alpha\Delta\xi + \cos\gamma\cos\beta\Delta\eta\big)\Big]\Big\}. \end{cases}$$

• ¶ 7. The formulation of accelerate force.

$$(18)_C \quad \begin{cases} \Delta\xi = r\Big(\frac{\partial\xi}{\partial a}\cos\alpha + \frac{\partial\xi}{\partial b}\cos\beta + \frac{\partial\xi}{\partial c}\cos\gamma\Big) \\ +\frac{r^2}{1\cdot 2}\Big(\frac{\partial^2\xi}{\partial a^2}\cos^2\alpha + \frac{\partial^2\xi}{\partial b^2}\cos^2\beta + \frac{\partial^2\xi}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\xi}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\xi}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\xi}{\partial a\partial b}\cos\alpha\cos\beta\Big) \\ +\cdots, \\ \Delta\eta = r\Big(\frac{\partial\eta}{\partial a}\cos\alpha + \frac{\partial\eta}{\partial b}\cos\beta + \frac{\partial\eta}{\partial c}\cos\gamma\Big) \\ +\frac{r^2}{1\cdot 2}\Big(\frac{\partial^2\eta}{\partial a^2}\cos^2\alpha + \frac{\partial^2\eta}{\partial b^2}\cos^2\beta + \frac{\partial^2\eta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\eta}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\eta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\eta}{\partial a\partial b}\cos\alpha\cos\beta\Big) \\ +\cdots, \\ \Delta\zeta = r\Big(\frac{\partial\zeta}{\partial a}\cos\alpha + \frac{\partial\zeta}{\partial b}\cos\beta + \frac{\partial\zeta}{\partial c}\cos\gamma\Big) \\ +\frac{r^2}{1\cdot 2}\Big(\frac{\partial^2\zeta}{\partial a^2}\cos^2\alpha + \frac{\partial^2\zeta}{\partial b^2}\cos^2\beta + \frac{\partial^2\zeta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\zeta}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\zeta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\zeta}{\partial a\partial b}\cos\alpha\cos\beta\Big) \\ +\cdots, \end{cases}$$

$$(19)_C \quad \frac{\partial\xi}{\partial a},\ \frac{\partial\xi}{\partial b},\ \frac{\partial\xi}{\partial c},\ \frac{\partial\eta}{\partial a},\ \frac{\partial\eta}{\partial b},\ \frac{\partial\eta}{\partial c},\ \frac{\partial\zeta}{\partial a},\ \frac{\partial\zeta}{\partial b},\ \frac{\partial\zeta}{\partial c},$$

$$(20)_C \quad \begin{cases} \frac{\partial^2\xi}{\partial a^2},\ \frac{\partial^2\xi}{\partial b^2},\ \frac{\partial^2\xi}{\partial c^2},\ \frac{\partial^2\xi}{\partial b\partial c},\ \frac{\partial^2\xi}{\partial c\partial a},\ \frac{\partial^2\xi}{\partial a\partial b}; \\ \frac{\partial^2\eta}{\partial a^2},\ \frac{\partial^2\eta}{\partial b^2},\ \frac{\partial^2\eta}{\partial c^2},\ \frac{\partial^2\eta}{\partial b\partial c},\ \frac{\partial^2\eta}{\partial c\partial a},\ \frac{\partial^2\eta}{\partial a\partial b}, \\ \frac{\partial^2\zeta}{\partial a^2},\ \frac{\partial^2\zeta}{\partial b^2},\ \frac{\partial^2\zeta}{\partial c^2},\ \frac{\partial^2\xi}{\partial b\partial c},\ \frac{\partial^2\zeta}{\partial c\partial a},\ \frac{\partial^2\zeta}{\partial a\partial b}, \end{cases}$$

We show ξ_1, η_1, ζ_1 with Jacobian :

$$(21)_C \quad \begin{cases} \xi_1 = \frac{\partial\xi}{\partial a}\cos\alpha + \frac{\partial\xi}{\partial b}\cos\beta + \frac{\partial\xi}{\partial c}\cos\gamma, \\ \eta_1 = \frac{\partial\eta}{\partial a}\cos\alpha + \frac{\partial\eta}{\partial b}\cos\beta + \frac{\partial\eta}{\partial c}\cos\gamma, \\ \zeta_1 = \frac{\partial\zeta}{\partial a}\cos\alpha + \frac{\partial\zeta}{\partial b}\cos\beta + \frac{\partial\zeta}{\partial c}\cos\gamma. \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{bmatrix} = \begin{bmatrix} \frac{\partial\xi}{\partial a} & \frac{\partial\xi}{\partial b} & \frac{\partial\xi}{\partial c} \\ \frac{\partial\eta}{\partial a} & \frac{\partial\eta}{\partial b} & \frac{\partial\eta}{\partial c} \\ \frac{\partial\zeta}{\partial a} & \frac{\partial\zeta}{\partial b} & \frac{\partial\zeta}{\partial c} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$

$$(22)_C \quad \begin{cases} \xi_2 = \frac{\partial^2\xi}{\partial a^2}\cos^2\alpha + \frac{\partial^2\xi}{\partial b^2}\cos^2\beta + \frac{\partial^2\xi}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\xi}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\xi}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\xi}{\partial a\partial b}\cos\alpha\cos\beta, \\ \eta_2 = \frac{\partial^2\eta}{\partial a^2}\cos^2\alpha + \frac{\partial^2\eta}{\partial b^2}\cos^2\beta + \frac{\partial^2\eta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\eta}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\eta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\eta}{\partial a\partial b}\cos\alpha\cos\beta, \\ \zeta_2 = \frac{\partial^2\zeta}{\partial a^2}\cos^2\alpha + \frac{\partial^2\zeta}{\partial b^2}\cos^2\beta + \frac{\partial^2\zeta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\xi}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\zeta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\zeta}{\partial a\partial b}\cos\alpha\cos\beta, \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \frac{\partial^2\xi}{\partial b^2} & \frac{\partial^2\xi}{\partial c^2} & \frac{\partial^2\xi}{\partial b\partial c} & \frac{\partial^2\xi}{\partial c\partial a} & \frac{\partial^2\xi}{\partial a\partial b} \\ \frac{\partial^2\eta}{\partial a^2} & \frac{\partial^2\eta}{\partial b^2} & \frac{\partial^2\eta}{\partial c^2} & \frac{\partial^2\eta}{\partial b\partial c} & \frac{\partial^2\eta}{\partial c\partial a} & \frac{\partial^2\eta}{\partial a\partial b} \\ \frac{\partial^2\zeta}{\partial a^2} & \frac{\partial^2\zeta}{\partial b^2} & \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\xi}{\partial b\partial c} & \frac{\partial^2\zeta}{\partial c\partial a} & \frac{\partial^2\zeta}{\partial a\partial b} \end{bmatrix} \begin{bmatrix} \cos^2\alpha \\ \cos^2\beta \\ \cos^2\gamma \\ 2\cos\beta\cos\gamma \\ 2\cos\gamma\cos\alpha \\ 2\cos\alpha\cos\beta \end{bmatrix}$$

From $(18)_C$, we get the following :

$$(23)_C \quad \Delta\xi = r\Big(\xi_1 + \frac{r}{2}\xi_2\Big), \quad \Delta\eta = r\Big(\eta_1 + \frac{r}{2}\eta_2\Big), \quad \Delta\zeta = r\Big(\zeta_1 + \frac{r}{2}\zeta_2\Big),$$

and from $(12)_C$ and $(23)_C$, we get the following :

$$(24)_C \quad \varepsilon = \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) = \xi_1\cos\alpha + \eta_1\cos\beta + \zeta_1\cos\gamma + \frac{r}{2}\Big(\xi_2\cos\alpha + \eta_2\cos\beta + \zeta_2\cos\gamma\Big)$$

The equation $(14)_C$ turns into as follows :

$$(14)_C \quad \Rightarrow \quad (25)_C \quad X = X_0 + X_1 + X_2, \quad Y = Y_0 + Y_1 + Y_2, \quad Z = Z_0 + Z_1 + Z_2$$

$$(26)_C \quad X_0 = \mathbf{S}[\pm m\cos\alpha\mathrm{f}(r)], \quad Y_0 = \mathbf{S}[\pm m\cos\beta\mathrm{f}(r)], \quad Z_0 = \mathbf{S}[\pm m\cos\gamma\mathrm{f}(r)]$$

$$(27)_C \left\{\begin{array}{l} X_1 = \mathbf{S}[\pm m\xi_1 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm m(\xi_1 \cos\alpha + \eta_1 \cos\beta + \zeta_1 \cos\gamma)\cos\alpha[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Y_1 = \mathbf{S}[\pm m\eta_1 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm m(\xi_1 \cos\alpha + \eta_1 \cos\beta + \zeta_1 \cos\gamma)\cos\beta[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Z_1 = \mathbf{S}[\pm m\zeta_1 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm m(\xi_1 \cos\alpha + \eta_1 \cos\beta + \zeta_1 \cos\gamma)\cos\gamma[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \end{array}\right.$$

$$(28)_C \left\{\begin{array}{l} X_2 = \mathbf{S}[\pm \frac{mr}{2}\xi_2 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm \frac{mr}{2}(\xi_2 \cos\alpha + \eta_2 \cos\beta + \zeta_2 \cos\gamma)\cos\alpha[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Y_2 = \mathbf{S}[\pm \frac{mr}{2}\eta_2 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm \frac{mr}{2}(\xi_2 \cos\alpha + \eta_2 \cos\beta + \zeta_2 \cos\gamma)\cos\beta[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Z_2 = \mathbf{S}[\pm \frac{mr}{2}\zeta_2 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm \frac{mr}{2}(\xi_2 \cos\alpha + \eta_2 \cos\beta + \zeta_2 \cos\gamma)\cos\gamma[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \end{array}\right.$$

We put $f(r)$ in brief as follows :

$$(29)_C \quad f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)] \tag{83}$$

$$(30)_C \left\{\begin{array}{l} X_1 = X_0\frac{\partial\xi}{\partial a} + Y_0\frac{\partial\xi}{\partial b} + Z_0\frac{\partial\xi}{\partial c} \\ \quad + \frac{\partial\xi}{\partial a}\mathbf{S}[mf(r)\cos^3\alpha] + \frac{\partial\xi}{\partial b}\mathbf{S}[mf(r)\cos^2\alpha\cos\beta] + \frac{\partial\xi}{\partial c}\mathbf{S}[mf(r)\cos^2\alpha\cos\gamma] \\ \quad + \frac{\partial\eta}{\partial a}\mathbf{S}[mf(r)\cos^2\alpha\cos\beta] + \frac{\partial\eta}{\partial b}\mathbf{S}[mf(r)\cos\alpha\cos^2\beta] + \frac{\partial\eta}{\partial c}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma] \\ \quad + \frac{\partial\zeta}{\partial a}\mathbf{S}[mf(r)\cos^2\alpha\cos\gamma] + \frac{\partial\zeta}{\partial b}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma] + \frac{\partial\zeta}{\partial c}\mathbf{S}[mf(r)\cos\alpha\cos^2\gamma], \\ Y_1 = X_0\frac{\partial\eta}{\partial a} + Y_0\frac{\partial\eta}{\partial b} + Z_0\frac{\partial\eta}{\partial c} \\ \quad + \frac{\partial\xi}{\partial a}\mathbf{S}[mf(r)\cos^2\alpha\cos\beta] + \frac{\partial\xi}{\partial b}\mathbf{S}[mf(r)\cos\alpha\cos^2\beta] + \frac{\partial\xi}{\partial c}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma] \\ \quad + \frac{\partial\eta}{\partial a}\mathbf{S}[mf(r)\cos\alpha\cos^2\beta] + \frac{\partial\eta}{\partial b}\mathbf{S}[mf(r)\cos^3\beta] + \frac{\partial\eta}{\partial c}\mathbf{S}[mf(r)\cos^2\beta\cos\gamma] \\ \quad + \frac{\partial\zeta}{\partial a}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma] + \frac{\partial\zeta}{\partial b}\mathbf{S}[mf(r)\cos^2\beta\cos\gamma] + \frac{\partial\zeta}{\partial c}\mathbf{S}[mf(r)\cos\beta\cos^2\gamma], \\ Z_1 = X_0\frac{\partial\zeta}{\partial a} + Y_0\frac{\partial\zeta}{\partial b} + Z_0\frac{\partial\zeta}{\partial c} + \cdots \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} X_1 = mf(r) \begin{bmatrix} \frac{\partial\xi}{\partial a}\big(\frac{X_0}{mf(r)\cos\alpha} + \mathbf{S}\cos^2\alpha\big) & \frac{\partial\xi}{\partial b}\mathbf{S}\cos^2\alpha & \frac{\partial\xi}{\partial c}\mathbf{S}\cos^2\alpha \\ \frac{\partial\eta}{\partial a}\big(\frac{Y_0}{mf(r)\cos\alpha} + \mathbf{S}\cos\alpha\cos\beta\big) & \frac{\partial\eta}{\partial b}\mathbf{S}\cos\alpha\cos\beta & \frac{\partial\eta}{\partial c}\mathbf{S}\cos\alpha\cos\beta \\ \frac{\partial\zeta}{\partial a}\big(\frac{Z_0}{mf(r)\cos\alpha} + \mathbf{S}\cos\alpha\cos\gamma\big) & \frac{\partial\zeta}{\partial b}\mathbf{S}\cos\alpha\cos\gamma & \frac{\partial\zeta}{\partial c}\mathbf{S}\cos\alpha\cos\gamma \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} \\ Y_1 = mf(r) \begin{bmatrix} \frac{\partial\xi}{\partial a}\mathbf{S}\cos\alpha\cos\beta & \frac{\partial\xi}{\partial b}\big(\frac{X_0}{mf(r)\cos\beta} + \mathbf{S}\cos\alpha\cos\beta\big) & \frac{\partial\xi}{\partial c}\mathbf{S}\cos\alpha\cos\beta \\ \frac{\partial\eta}{\partial a}\mathbf{S}\cos^2\beta & \frac{\partial\eta}{\partial b}\big(\frac{Y_0}{mf(r)\cos\beta} + \mathbf{S}\cos^2\beta\big) & \frac{\partial\eta}{\partial c}\mathbf{S}\cos^2\beta \\ \frac{\partial\zeta}{\partial a}\mathbf{S}\cos\beta\cos\gamma & \frac{\partial\zeta}{\partial b}\big(\frac{Z_0}{mf(r)\cos\beta} + \mathbf{S}\cos\beta\cos\gamma\big) & \frac{\partial\zeta}{\partial c}\mathbf{S}\cos\beta\cos\gamma \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} \\ Z_1 = mf(r) \begin{bmatrix} \frac{\partial\xi}{\partial a}\mathbf{S}\cos\alpha\cos\gamma & \frac{\partial\xi}{\partial b}\mathbf{S}\cos\alpha\cos\gamma & \frac{\partial\xi}{\partial c}\big(\frac{X_0}{mf(r)\cos\gamma} + \mathbf{S}\cos\alpha\cos\gamma\big) \\ \frac{\partial\eta}{\partial a}\mathbf{S}\cos\beta\cos\gamma & \frac{\partial\eta}{\partial b}\mathbf{S}\cos\beta\cos\gamma & \frac{\partial\eta}{\partial c}\big(\frac{Y_0}{mf(r)\cos\gamma} + \mathbf{S}\cos\beta\cos\gamma\big) \\ \frac{\partial\zeta}{\partial a}\mathbf{S}\cos^2\gamma & \frac{\partial\zeta}{\partial b}\mathbf{S}\cos^2\gamma & \frac{\partial\zeta}{\partial c}\big(\frac{Z_0}{mf(r)\cos\gamma} + \mathbf{S}\cos^2\gamma\big) \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} \end{array}\right.$$

We see X, Y and Z are computed in according to $(25)_C$ by only X_2, Y_2, Z_2, because in $(26)_C, (27)_C$ and $(30)_C$, all the terms contain the terms of $\cos\alpha$ or $\cos\beta$ or $\cos\gamma$ in odd power, which become zero in summation under the symbol of $\mathbf{\$}$.

$$
(31)_C \quad \begin{cases}
X_2 = \frac{\partial^2\xi}{\partial a^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \frac{\partial^2\xi}{\partial b^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \frac{\partial^2\xi}{\partial c^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] \\
\quad + \ \frac{\partial^2\xi}{\partial b\partial c}\ \mathbf{\$}[\pm mr\cos\beta\cos\gamma \mathrm{f}(r)] + \frac{\partial^2\xi}{\partial c\partial a}\ \mathbf{\$}[\pm mr\cos\gamma\cos\alpha \mathrm{f}(r)] + \frac{\partial^2\xi}{\partial a\partial b}\ \mathbf{\$}[\pm mr\cos\alpha\cos\beta \mathrm{f}(r)] \\
\quad + \ \frac{\partial^2\xi}{\partial a^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^4\alpha] + \frac{\partial^2\xi}{\partial b^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\xi}{\partial c^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\gamma] \\
\quad + \ \frac{\partial^2\xi}{\partial b\partial c}\ \mathbf{\$}[mrf(r)\cos^2\alpha\cos\beta\cos\gamma] + \frac{\partial^2\xi}{\partial c\partial a}\ \mathbf{\$}[mrf(r)\cos^3\alpha\cos\gamma] + \frac{\partial^2\xi}{\partial a\partial b}\ \mathbf{\$}[mrf(r)\cos^3\alpha\cos\beta] \\
\quad + \ \frac{\partial^2\eta}{\partial a^2}\ \mathbf{\$}[\frac{mr}{2}f(r)\cos^3\alpha\cos\beta] + \frac{\partial^2\eta}{\partial b^2}\ \mathbf{\$}[\frac{mr}{2}f(r)\cos\alpha\cos^3\beta] + \frac{\partial^2\eta}{\partial c^2}\ \mathbf{\$}[\frac{mr}{2}f(r)\cos\alpha\cos\beta\cos^2\gamma] \\
\quad + \ \frac{\partial^2\eta}{\partial b\partial c}\ \mathbf{\$}[mrf(r)\cos\alpha\cos^2\beta\cos\gamma] + \ \frac{\partial^2\eta}{\partial a\partial b}\ \mathbf{\$}[mrf(r)\cos^2\alpha\cos\beta\cos\gamma] + \ \frac{\partial^2\eta}{\partial a\partial b}\ \mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta] \\
\quad + \ \frac{\partial^2\zeta}{\partial a^2}\ \mathbf{\$}[\frac{mr}{2}f(r)\cos^3\alpha\cos\gamma] + \frac{\partial^2\zeta}{\partial b^2}\ \mathbf{\$}[\frac{mr}{2}f(r)\cos\alpha\cos^2\beta\cos\gamma] + \frac{\partial^2\zeta}{\partial c^2}\ \mathbf{\$}[\frac{mr}{2}f(r)\cos\alpha\cos^3\gamma] \\
\quad + \ \frac{\partial^2\zeta}{\partial b\partial c}\ \mathbf{\$}[mrf(r)\cos\alpha\cos\beta\cos^2\gamma] + \frac{\partial^2\zeta}{\partial c\partial a}\ \mathbf{S}[mrf(r)\cos^2\alpha\cos^2\gamma] + \frac{\partial^2\zeta}{\partial a\partial b}\ \mathbf{\$}[mrf(r)\cos^2\alpha\cos\beta\cos\gamma], \\
Y_2 = \frac{\partial^2\eta}{\partial a^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \frac{\partial^2\eta}{\partial b^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \frac{\partial^2\eta}{\partial c^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] \\
\quad + \cdots \\
\quad + \frac{\partial^2\eta}{\partial a^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\eta}{\partial b^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^4\beta] + \frac{\partial^2\eta}{\partial c^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma] \\
\quad + \cdots \\
\quad + \frac{\partial^2\xi}{\partial a\partial b}\ \mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta] \\
\quad + \cdots \\
\quad + \frac{\partial^2\zeta}{\partial b\partial c}\ \mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma], \\
Z_2 = \frac{\partial^2\zeta}{\partial a^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial b^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial c^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] \\
\quad + \cdots \\
\quad + \frac{\partial^2\zeta}{\partial a^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\xi}{\partial b^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma] + \frac{\partial^2\zeta}{\partial c^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^4\gamma] \\
\quad + \cdots \\
\quad + \frac{\partial^2\xi}{\partial c\partial c}\ \mathbf{S}[mrf(r)\cos^2\gamma\cos^2\alpha] \\
\quad + \cdots \\
\quad + \frac{\partial^2\eta}{\partial b\partial c}\ \mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma]
\end{cases}
$$

• ¶ 8. Remark : in the right-hand sides of $(25)_C$, $X_2,\ Y_2,\ Z_2$ are not only the largest valued terms in $(25)_C$, but also non-zero terms in $(25)_C$, due to the same signe.

• ¶ 9. Remark : the equations of accelerate forces follow not only in the forces come from its mutual attraction or repulsion but also in the new accelerate forces. [25]

(#6). (Omitted.)

$$(32)_C \quad X + \mathcal{X} = 0, \quad Y + \mathcal{Y} = 0, \quad Z + \mathcal{Z} = 0.$$

Au contraire, si le systéme se meut, en désignant par ψ la force accélératrice qui serait capable de produire à elle seule le mouvement effectif de la molécule m, et par $\dot{X},\ \dot{Y},\ \dot{Z}$ les projections algébriques de cette force sur les axes coordonnées, on devra, dans les équations $(32)_C$, remlacer les quantités $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ par les différences $\mathcal{X}-\dot{X},\ \mathcal{Y}-\dot{Y},\ \mathcal{Z}-\dot{Z}$. Comme on trouvera, d'ailleurs, en prenant $a,\ b,\ c$ pour variables indépendantes, et ayant égard aux formules $(4)_C$, [6, pp.236-7]

$$(4)_C \quad x = a + \xi, \quad y = b + \eta, \quad z = c + \zeta.$$

$$(33)_C \quad \dot{X} = \frac{\partial^2 x}{\partial t^2} = \frac{\partial^2\xi}{\partial t^2}, \quad \dot{Y} = \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2\eta}{\partial t^2}, \quad \dot{Z} = \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2\zeta}{\partial t^2},$$

il est clair que le mouvement d'un molécule quelconque m sera déterminé par les équations

Replacing $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ of $(32)_C$ with $\mathcal{X}-\dot{X},\ \mathcal{Y}-\dot{Y},\ \mathcal{Z}-\dot{Z}$, and considering $(4)_C$, we get the new accelerate forces as follows :

$$(32)_C \quad \Rightarrow \quad \begin{cases} X + \mathcal{X} - \dot{X} = 0, \\ Y + \mathcal{Y} - \dot{Y} = 0, \\ Z + \mathcal{Z} - \dot{Z} = 0, \end{cases} \quad \Rightarrow \quad (34)_C \quad \begin{cases} X + \mathcal{X} = \dot{X} = \frac{\partial^2\xi}{\partial t^2}, \\ Y + \mathcal{Y} = \dot{Y} = \frac{\partial^2\eta}{\partial t^2}, \\ Z + \mathcal{Z} = \dot{Z} = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$$

[25] These analysis is not appear in Navier.

3.3.2. Reduction of tensor.

• ¶ 10. The values of X, Y and Z , determinated by the statements $(25)_C$, $(26)_C$, $(30)_C$, $(31)_C$, are simplified with several hypothesis as follows :

• ¶ 11. $(26)_C$ and $(30)_C$ are disappear and X, Y, Z are reduced to only X_2, Y_2, Z_2 for the symmetric distribution of the molecules.

(#7). (Omitted.)

> Alors les formules $(15)_C$ seront vérifiées, c'est-à-dire que l'etat primitif du système sera un état d'equilibre ; et, comm on aura d'alilleurs
>
> $$(35)_C \quad X_1 = 0,\ Y_1 = 0,\ Z_1 = 0,$$
>
> les valeurs de X, Y, Z se réduiront à celles de X_2, Y_2, Z_2. [6, pp.237-8]

• ¶ 12. At last, we get X, Y and Z from $(26)_C$, $(30)_C$, $(31)_C$ after deleting the terms containing the terms of $\cos\alpha$ or $\cos\beta$ or $\cos\gamma$ in odd power, which becomes zero in summation, (ex. including $\cos\alpha$, $\cos^3\alpha$, $\cos\beta$, $\cos^3\beta$, $\cdots$) (#8). (Omitted.)

$$(36)_C \quad \begin{cases} X = \frac{\partial^2\xi}{\partial a^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \frac{\partial^2\xi}{\partial b^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \frac{\partial^2\xi}{\partial c^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)] \\ \quad + \frac{\partial^2\xi}{\partial a^2}\mathbf{S}[\frac{mr}{2}\cos^4\alpha f(r)] + \frac{\partial^2\xi}{\partial b^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\xi}{\partial c^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\gamma f(r)] \\ \quad + \frac{\partial^2\eta}{\partial a\partial b}\mathbf{S}[mr\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\zeta}{\partial a\partial c}\mathbf{S}[mr\cos^2\alpha\cos^2\gamma f(r)], \\ Y = \frac{\partial^2\eta}{\partial a^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \frac{\partial^2\eta}{\partial b^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \frac{\partial^2\eta}{\partial c^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)] \\ \quad + \frac{\partial^2\eta}{\partial a^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\eta}{\partial b^2}\mathbf{S}[\frac{mr}{2}\cos^4\beta f(r)] + \frac{\partial^2\eta}{\partial c^2}\mathbf{S}[\frac{mr}{2}\cos^2\beta\cos^2\gamma f(r)] \\ \quad + \frac{\partial^2\xi}{\partial a\partial b}\mathbf{S}[mr\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\zeta}{\partial b\partial c}\mathbf{S}[mr\cos^2\beta\cos^2\gamma f(r)], \\ Z = \frac{\partial^2\zeta}{\partial a^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial b^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial c^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)] \\ \quad + \frac{\partial^2\zeta}{\partial a^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\zeta}{\partial b^2}\mathbf{S}[\frac{mr}{2}\cos^2\beta\cos^2\gamma f(r)] + \frac{\partial^2\zeta}{\partial c^2}\mathbf{S}[\frac{mr}{2}\cos^4\gamma f(r)] \\ \quad + \frac{\partial^2\xi}{\partial c\partial a}\mathbf{S}[mr\cos^2\gamma\cos^2\alpha f(r)] + \frac{\partial^2\eta}{\partial b\partial c}\mathbf{S}[mr\cos^2\beta\cos^2\gamma f(r)] \end{cases}$$

$\Rightarrow$

$$\begin{cases} X = \frac{\partial^2\xi}{\partial a^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^4\alpha]\right\} \\ \quad + \frac{\partial^2\xi}{\partial b^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta]\right\} \\ \quad + \frac{\partial^2\xi}{\partial c^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\gamma]\right\} \\ \quad + \frac{\partial^2\eta}{\partial a\partial b}\mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\zeta}{\partial c\partial a}\mathbf{S}[mrf(r)\cos^2\alpha\cos^2\gamma], \\ Y = \frac{\partial^2\eta}{\partial a^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta]\right\} \\ \quad + \frac{\partial^2\eta}{\partial b^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^4\beta]\right\} \\ \quad + \frac{\partial^2\eta}{\partial c^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma]\right\} \\ \quad + \frac{\partial^2\zeta}{\partial b\partial c}\mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma] + \frac{\partial^2\xi}{\partial a\partial b}\mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta], \\ Z = \frac{\partial^2\zeta}{\partial a^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta]\right\} \\ \quad + \frac{\partial^2\zeta}{\partial b^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma]\right\} \\ \quad + \frac{\partial^2\zeta}{\partial c^2}\left\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^4\gamma]\right\} \\ \quad + \frac{\partial^2\xi}{\partial c\partial a}\mathbf{S}[mrf(r)\cos^2\gamma\cos^2\alpha] + \frac{\partial^2\eta}{\partial b\partial c}\mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma] \end{cases} \tag{84}$$

$$\Rightarrow \quad \begin{bmatrix} X & Y & Z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \frac{\partial^2\xi}{\partial b^2} & \frac{\partial^2\xi}{\partial c^2} & \frac{\partial^2\eta}{\partial a\partial b} & \frac{\partial^2\zeta}{\partial c\partial a} \\ \frac{\partial^2\eta}{\partial a^2} & \frac{\partial^2\eta}{\partial b^2} & \frac{\partial^2\eta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b\partial c} & \frac{\partial^2\xi}{\partial a\partial b} \\ \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b^2} & \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\xi}{\partial c\partial a} & \frac{\partial^2\eta}{\partial b\partial c} \end{bmatrix}$$

$$\times \quad \mathbf{S}\frac{mr}{2} \begin{bmatrix} \pm\cos^2\alpha\mathrm{f}(r) + f(r)\cos^4\alpha & \pm\cos^2\alpha\mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\beta & \pm\cos^2\alpha\mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\beta \\ \pm\cos^2\beta\mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\beta & \pm\cos^2\beta\mathrm{f}(r) + f(r)\cos^4\beta & \pm\cos^2\beta\mathrm{f}(r) + f(r)\cos^2\beta\cos^2\gamma \\ \pm\cos^2\gamma\mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\gamma & \pm\cos^2\gamma\mathrm{f}(r) + f(r)\cos^2\beta\cos^2\gamma & \pm\cos^2\gamma\mathrm{f}(r) + f(r)\cos^4\gamma \\ 2f(r)\cos^2\alpha\cos^2\beta & 2f(r)\cos^2\beta\cos^2\gamma & 2f(r)\cos^2\gamma\cos^2\alpha \\ 2f(r)\cos^2\alpha\cos^2\gamma & 2f(r)\cos^2\alpha\cos^2\beta & 2f(r)\cos^2\beta\cos^2\gamma \end{bmatrix}$$

$$\equiv \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \frac{\partial^2\xi}{\partial b^2} & \frac{\partial^2\xi}{\partial c^2} & \frac{\partial^2\eta}{\partial a\partial b} & \frac{\partial^2\zeta}{\partial c\partial a} \\ \frac{\partial^2\eta}{\partial a^2} & \frac{\partial^2\eta}{\partial b^2} & \frac{\partial^2\eta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b\partial c} & \frac{\partial^2\xi}{\partial a\partial b} \\ \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b^2} & \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\xi}{\partial c\partial a} & \frac{\partial^2\eta}{\partial b\partial c} \end{bmatrix} \begin{bmatrix} G+L & G+R & G+Q \\ H+P & H+M & H+P \\ I+Q & I+P & I+N \\ 2R & 2P & 2Q \\ 2Q & 2R & 2P \end{bmatrix},$$

where, we define 9 parameters in (84) by G, H, I, L, M, N, P, Q and R as follows :

$$\begin{cases} (37)_C \quad G \equiv \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\ \mathrm{f}(r)], \quad H \equiv \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\ \mathrm{f}(r)], \quad I \equiv \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\ \mathrm{f}(r)], \\ (38)_C \quad L \equiv \mathbf{S}[\frac{mr}{2}\cos^4\alpha\ f(r)], \quad M \equiv \mathbf{S}[\frac{mr}{2}\cos^4\beta\ f(r)], \quad N \equiv \mathbf{S}[\frac{mr}{2}\cos^4\gamma\ f(r)], \\ (39)_C \quad P \equiv \mathbf{S}[\frac{mr}{2}\cos^2\beta\cos^2\gamma\ f(r)], \quad Q \equiv \mathbf{S}[\frac{mr}{2}\cos^2\gamma\cos^2\alpha\ f(r)], \quad R \equiv \mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta\ f(r)], \end{cases}$$

Then from (84) it turns into as follows :

$$(40)_C \quad \begin{cases} X = (G+L)\frac{\partial^2\xi}{\partial a^2} + (H+R)\frac{\partial^2\xi}{\partial b^2} + (I+Q)\frac{\partial^2\xi}{\partial c^2} + 2R\frac{\partial^2\eta}{\partial a\partial b} + 2Q\frac{\partial^2\zeta}{\partial c\partial a}, \\ Y = (G+R)\frac{\partial^2\eta}{\partial a^2} + (H+M)\frac{\partial^2\eta}{\partial b^2} + (I+P)\frac{\partial^2\eta}{\partial c^2} + 2P\frac{\partial^2\zeta}{\partial b\partial c} + 2R\frac{\partial^2\xi}{\partial a\partial b}, \\ Z = (G+Q)\frac{\partial^2\zeta}{\partial a^2} + (H+P)\frac{\partial^2\zeta}{\partial b^2} + (I+N)\frac{\partial^2\zeta}{\partial c^2} + 2Q\frac{\partial^2\xi}{\partial c\partial a} + 2P\frac{\partial^2\eta}{\partial b\partial c}. \end{cases}$$

• ¶ 13. Invariable values : $G, H, I, L, M, N, P, Q, R$.

If we suppose that the molecules $m, m', m'', \cdots$ are originally distributed by the same way in relation to the three planes made by the molecule m in parallel with the plane coordinates, then the values of quantities: $G, H, I, L, M, N, P, Q, R$ come to remain invariable, even though a series of changes are made among the three angles : α, β, γ.

(#9). (Omitted.)

$$(41)_C \quad G = H = I, \quad L = M = N, \quad P = Q = R.$$

$$(42)_C \quad \begin{cases} X = (L+G)\frac{\partial^2\xi}{\partial a^2} + (R+G)\Big(\frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\Big) + 2R\Big(\frac{\partial^2\eta}{\partial a\partial b} + \frac{\partial^2\zeta}{\partial c\partial a}\Big), \\ Y = (L+G)\frac{\partial^2\eta}{\partial b^2} + (R+G)\Big(\frac{\partial^2\eta}{\partial c^2} + \frac{\partial^2\eta}{\partial a^2}\Big) + 2R\Big(\frac{\partial^2\zeta}{\partial b\partial c} + \frac{\partial^2\xi}{\partial a\partial b}\Big), \\ Z = (L+Q)\frac{\partial^2\zeta}{\partial c^2} + (R+G)\Big(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2}\Big) + 2R\Big(\frac{\partial^2\xi}{\partial c\partial a} + \frac{\partial^2\eta}{\partial b\partial c}\Big). \end{cases} \qquad (85)$$

$$\Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \Big(\frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\Big) & \Big(\frac{\partial^2\eta}{\partial a\partial b} + \frac{\partial^2\zeta}{\partial c\partial a}\Big) \\ \frac{\partial^2\eta}{\partial a^2} & \Big(\frac{\partial^2\eta}{\partial b^2} + \frac{\partial^2\eta}{\partial c^2}\Big) & \Big(\frac{\partial^2\zeta}{\partial b\partial c} + \frac{\partial^2\xi}{\partial a\partial b}\Big) \\ \frac{\partial^2\zeta}{\partial c^2} & \Big(\frac{\partial^2\zeta}{\partial b^2} + \frac{\partial^2\zeta}{\partial c^2}\Big) & \Big(\frac{\partial^2\xi}{\partial c\partial a} + \frac{\partial^2\eta}{\partial b\partial c}\Big) \end{bmatrix} \begin{bmatrix} L+G \\ R+G \\ 2R \end{bmatrix}.$$

• ¶ 14. For the angles : $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$ are parpendicular among each planes, the values of sums : $G, H, I, L, M, N, P, Q, R$ does not alter even by replacing $\cos\alpha, \cos\beta, \cos\gamma$ with the trinomial : (#10). (Omitted.)

$$\begin{cases} \cos\alpha \ \Rightarrow\ \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1, \\ \cos\beta \ \Rightarrow\ \cos\alpha\cos\alpha_2 + \cos\beta\cos\beta_2 + \cos\gamma\cos\gamma_2, \\ \cos\gamma \ \Rightarrow\ \cos\alpha\cos\alpha_3 + \cos\beta\cos\beta_3 + \cos\gamma\cos\gamma_3, \end{cases}$$

$$(43)_C \quad \begin{cases} G = \mathbf{S}\Big[\pm\frac{mr}{2}\big(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1\big)^2\mathrm{f}(r)\Big], \\ L = \mathbf{S}\Big[\frac{mr}{2}\big(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1\big)^4 f(r)\Big], \\ R = \mathbf{S}\Big[\frac{mr}{2}\big(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1\big)^2\big(\cos\alpha\cos\alpha_2 + \cos\beta\cos\beta_2 + \cos\gamma\cos\gamma_2\big)^2 f(r)\Big] \end{cases} \qquad (86)$$

$(44)_C$

$$\begin{cases} G = G\big(\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1\big) \equiv\ GA_1, \\ L = L\big(\cos^4\alpha_1 + \cos^4\beta_1 + \cos^4\gamma_1\big) + 6R\big(\cos^2\beta_1\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_1\big) \equiv LB + 6RC, \\ R = R\big(\cos^2\beta_1\cos^2\gamma_2 + \cos^2\beta_2\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_2 + \cos^2\gamma_2\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_2 + \cos^2\alpha_2\cos^2\beta_1\big), \quad (8 \\ \quad +4R\big(\cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2 + \cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2 + \cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2\big) \\ \quad +L\big(\cos^2\alpha_1\cos^2\alpha_2 + \cos^2\beta_1\cos^2\beta_2 + \cos^2\gamma_1\cos^2\gamma_2\big) \equiv RD + 4RE + LF, \end{cases}$$

where

$$\begin{cases}\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1=1,\\ \cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2=1,\\ \cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0\end{cases}$$

and

$$\begin{cases}A_1\equiv\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1,\\ A_2\equiv\cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2,\\ B=\cos^4\alpha_1+\cos^4\beta_1+\cos^4\gamma_1,\\ C\equiv\cos^2\beta_1\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_1,\\ D\equiv\cos^2\beta_1\cos^2\gamma_2+\cos^2\beta_2\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_2+\cos^2\gamma_2\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1,\\ E\equiv\cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2+\cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2+\cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2,\\ F\equiv\cos^2\alpha_1\cos^2\alpha_2+\cos^2\beta_1\cos^2\beta_2+\cos^2\gamma_1\cos^2\gamma_2\end{cases}$$

then

$$\begin{cases}1-B=A_1^2-B=2C,\\ 1-D=A_1A_2-D=F=-2E,\end{cases}\tag{88}$$

namely :

$$\begin{cases}1-\left(\cos^4\alpha_1+\cos^4\beta_1+\cos^4\gamma_1\right)=\left(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1\right)^2-\left(\cos^4\alpha_1+\cos^4\beta_1+\cos^4\gamma_1\right)\\ \quad=2\left(\cos^2\beta_1\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_1\right),\\ 1-\left(\cos^2\beta_1\cos^2\gamma_2+\cos^2\beta_2\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_2+\cos^2\gamma_2\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1\right)\\ \quad=\left(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1\right)\left(\cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2\right)\\ \qquad-\left(\cos^2\beta_1\cos^2\gamma_2+\cos^2\beta_2\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_2+\cos^2\gamma_2\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1\right)\\ \quad=\left(\cos^2\alpha_1\cos^2\alpha_2+\cos^2\beta_1\cos^2\beta_2+\cos^2\gamma_1\cos^2\gamma_2\right)\\ \quad=\left(\cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2+\cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2+\cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2\right)\end{cases}$$

From the second equation of (87) (: $(44)_C$) by (88)

$$L(1-B)=2LC=6RC$$

$$(45)_C\quad L=3R,$$

or, from the third equation by (88)

$$R(1-D)=-2RE=4RE+LF\quad\Rightarrow 2LE=6RE$$

$$(45)_C\quad L=3R$$

From $(85)(=(42)_C)$ we get $(89)(=(46)_C)$ by $(45)_C$ as follows :

$$(46)_C\quad\begin{cases}X=(R+G)\left(\frac{\partial^2\xi}{\partial a^2}+\frac{\partial^2\xi}{\partial b^2}+\frac{\partial^2\xi}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial a},\\ Y=(R+G)\left(\frac{\partial^2\eta}{\partial a^2}+\frac{\partial^2\eta}{\partial b^2}+\frac{\partial^2\eta}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial b},\\ Z=(R+G)\left(\frac{\partial^2\zeta}{\partial a^2}+\frac{\partial^2\zeta}{\partial b^2}+\frac{\partial^2\zeta}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial c},\end{cases}\tag{89}$$

$$\text{where}\qquad (47)_C\quad \nu=\frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}.$$

(c.f. $(89)(=(46)_C)\Rightarrow(80)(=(7\text{-}9)_{Pf})\Rightarrow(97)(=(12)_S)$). Moreover, from $(41)_C$:

$$G=H=I,\quad L=M=N,\quad P=Q=R.$$

By the way, Cauchy says, when we put $G=H=I=0$ in $(40)_C$, we can see the coincidence of Cauchy's R with Navier's ε, as follows :

$$\begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix} = R \begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \tag{90}$$

These coefficients of (90) are equal to (16) of Navier. (c.f. Section 4. Concurrence, Table 13.)

• ¶ 15. Dencity : Δ defined by mass of a sphere : $\mathcal{M}$ and the volume of a sphere : $\mathcal{V}$ as follows :

Concevons maintenant que, dans l'etat primitif du système des molécules $m, m', m'', \cdots$, et, du point (a, b, c) comme centre avec un rayon l convenablement choisi, on décrive une sphère qui renferme toutes les molécules dont l'action sur la masse **m** à une valeur sensible. Divisions le volume $\mathcal{V}$ de cette sphère en éléments très petits $v, v', v'', \cdots$, mais dont chacun renferme encore un très grand nombre de molécules. Soient $\mathcal{M}$ la somme des masses des molécules comprises dans la sphère, et [6, p.241]

$$(48)_C \quad \Delta = \frac{\mathcal{M}}{\mathcal{V}} = \frac{\text{mass of system of particles}}{\text{volume of system of particles}} = \text{dencity}$$

(#11). Enfin supposons que les sommes des masses comprises les volemes élémentaires $v,\ v',\ v'', \cdots$ soient propotionelles à ces même volumes, et représentées en conséquence par les produits $\Delta v,\ \Delta v',\ \Delta v'', \cdots$. Alors, si la fonction f(r) est telle que, sans altérer sensiblement les sommes désignées par G et par R, on puisse faire abstraction de celles des molécules $m,\ m',\ m'', \cdots$ qui sont les plus voisines de la molécule m, les valeurs de G, R fournies par les équations $(37)_C$ et $(39)_C$ différeront très peu de celles que déterminent les formules

$$(49)_C \quad \begin{cases} G = \frac{\Delta}{2}\mathbf{S}[\pm r\cos^2\alpha \mathrm{f}(r)v], \\ R = \frac{\Delta}{2}\mathbf{S}[r\cos^2\alpha\cos^2\beta f(r)v] \end{cases}$$

quand on étend le signe **S**, non plus les points matériels $m,\ m',\ m'', \cdots$, mais à tous les éléments $v,\ v',\ v'', \cdots$ du volume $\mathcal{V}$.

Or, dans cette dernière hypothèse, le second membre de chacun des expression $(49)_C$ pourra être remplacé par une intégrale triple relative à trois coordonées polaires dont l'une serait le rayon vecteur **r**, tandis que les deux autres représenteraient les angles formés :

- par le rayon vecteur **r** avec l'axe des x ;
- par le plan qui renferme le même rayon et l'axe des x avec le plan des $x,\ y$.

[6, p.241-2]

Soient $p,\ q$ les deux angles dont il s'agit. Chaque intégrale triple devra être prise entre les limites $p = 0,\ q = \pi,\ q = 0,\ q = 2\pi,\ r = 0,\ r = l$; et l'on pourra même, sans erreur sensible, remplacer la seconde limite de r ou le rayon l par l'infini positif. [6, p.242]

$$(50)_C \quad \begin{cases} G = \pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3\mathrm{f}(r)\cos^2\alpha\sin p dr dq dp, \\ R = \frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\cos^2\beta\sin p dr dq dp \end{cases} \tag{91}$$

We compute in general case such that :

$$(51)_C \quad \begin{cases} \cos\alpha = \cos p, \\ \cos\beta = \sin p\cos q, \\ \cos\gamma = \sin p\sin q \end{cases}$$

$$\begin{cases} \int_0^{2\pi}\int_0^\pi \cos^2\alpha\sin p dq dp = 2\pi\int_0^\pi\cos^2 p\sin\ p dp = 2\pi\Big[-\frac{\cos^3 p}{3}\Big]_0^\pi = \frac{4\pi}{3}, \\ \int_0^{2\pi}\int_0^\pi\cos^2\alpha\cos^2\beta\sin\ p dp = \int_0^{2\pi}\cos^2 q dq\int_0^\pi\cos^2 p(1-\cos^2 p)\sin p dp \\ = \Big[\frac{q}{2}+\frac{1}{4}\sin 2q\Big]_0^{2\pi}\Big[-\frac{\cos^5 p}{5}\Big]_0^\pi = (\frac{2\pi}{2}-0)(\frac{2}{3}-\frac{2}{5}) = \frac{4\pi}{15} \end{cases} \tag{92}$$

$$C_3 = \frac{1}{2}\frac{4\pi}{15} = \frac{2\pi}{15}, \quad C_4 = \frac{1}{2}\frac{4\pi}{3} = \frac{2\pi}{3}$$

Then (91) turns out by (83) as follows :

$$(52)_C \quad \begin{cases} G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3 \mathrm{f}(r)dr, \\ R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr = \pm\frac{2\pi\Delta}{15}\int_0^\infty \left[r^4\mathrm{f}'(r) - r^3\mathrm{f}(r)\right]dr \end{cases} \qquad (93)$$

D'ailleurs, si, pour des valeurs croissantes de la distance r, la fonction $\mathrm{f}(r)$ décroit plus rapidment que la fonction que $\frac{1}{r^4}$, si de plus le produit $r^4\mathrm{f}(r)$ s'évanouit pour $r = 0$, on trouvera, en supposant la fonction $\mathrm{f}'(r)$ continue, et en intégrant par parties,

$$(53)_C \quad \int_0^\infty r^4\mathrm{f}'(r)dr = -4\int_0^\infty r^3\mathrm{f}(r)dr$$

On aura donc alors

$$(54)_C \quad R = -G,$$

et, par conséquent, on tirera des formules $(46)_C$

$$(55)_C \quad X = 2R\frac{\partial v}{\partial a}, \quad Y = 2R\frac{\partial v}{\partial b}, \quad Z = 2R\frac{\partial v}{\partial c}$$

• ¶ 16.

Lorsque les quantités, désinées dans les formules $(40)_C$ et $(48)_C$ par les lettres $G, H, I, L, M, N, P, Q, R$ et Δ, deviennent constantes, c'est-à-dire, indépendantes des coordonées $a,\ b,\ c$, ou, ce qui revient au même, de la place qu'occupe la molécule m, alors, en faisant, pour plus de commodité,

$$(56)_C \quad \begin{cases} A = \left[(L+G)\frac{\partial\xi}{\partial a} + (R-G)\frac{\partial\eta}{\partial b} + (Q-G)\frac{\partial\zeta}{\partial c}\right]\Delta, \\ B = \left[(R-H)\frac{\partial\xi}{\partial a} + (M+H)\frac{\partial\eta}{\partial b} + (P-H)\frac{\partial\zeta}{\partial c}\right]\Delta, \\ C = \left[(Q-I)\frac{\partial\xi}{\partial a} + (P-I)\frac{\partial\eta}{\partial b} + (N+I)\frac{\partial\zeta}{\partial c}\right]\Delta, \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \Delta \begin{bmatrix} L+G & R-G & Q-G \\ R-H & M+H & P-H \\ Q-I & P-I & N+I \end{bmatrix} \begin{bmatrix} \frac{\partial\xi}{\partial a} \\ \frac{\partial\eta}{\partial b} \\ \frac{\partial\zeta}{\partial c} \end{bmatrix}$$

$$(57)_C \quad \begin{cases} D = \left[(P+I)\frac{\partial\eta}{\partial c} + (P+H)\frac{\partial\zeta}{\partial b}\right]\Delta, \\ E = \left[(Q+G)\frac{\partial\zeta}{\partial a} + (Q+I)\frac{\partial\xi}{\partial c}\right]\Delta, \\ F = \left[(R+H)\frac{\partial\xi}{\partial b} + (R+G)\frac{\partial\eta}{\partial a}\right]\Delta, \end{cases} \quad \Rightarrow \quad \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \Delta \begin{bmatrix} 0 & P+I & P+H \\ Q+I & 0 & Q+G \\ R+H & R+G & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial\xi}{\partial c} & \frac{\partial\xi}{\partial b} \\ \frac{\partial\eta}{\partial c} & 0 & \frac{\partial\eta}{\partial a} \\ \frac{\partial\zeta}{\partial b} & \frac{\partial\zeta}{\partial a} & 0 \end{bmatrix}$$

We can reduce $(40)_C$ as follows :

$$(58)_C \quad \begin{cases} X = \frac{1}{\Delta}\left(\frac{\partial A}{\partial a} + \frac{\partial F}{\partial b} + \frac{\partial E}{\partial c}\right), \\ Y = \frac{1}{\Delta}\left(\frac{\partial F}{\partial a} + \frac{\partial B}{\partial b} + \frac{\partial D}{\partial c}\right), \\ X = \frac{1}{\Delta}\left(\frac{\partial E}{\partial a} + \frac{\partial D}{\partial b} + \frac{\partial C}{\partial c}\right), \end{cases} \quad \Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{\Delta} \left[\ \frac{\partial}{\partial a}\ \frac{\partial}{\partial b}\ \frac{\partial}{\partial c}\ \right] \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix}$$

By $(41)_C$ and $(45)_C$,

$$G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R.$$

$$\begin{aligned} \frac{A}{\Delta} &= (L+G)\frac{\partial\xi}{\partial a} + (R-G)\Big(\frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}\Big) \\ &= (L+G)\frac{\partial\eta}{\partial b} + (R-G)(v - \frac{\partial\zeta}{\partial c}) \\ &= (3R+G-R+G)\frac{\partial\xi}{\partial a} + (R-G)v \\ &= 2(R+G)\frac{\partial\xi}{\partial a} + (R-G)v \end{aligned}$$

$$\begin{aligned}\frac{B}{\Delta} = & \ (L+G)\frac{\partial\eta}{\partial b} + (R-G)\Big(\frac{\partial\xi}{\partial a} + \frac{\partial\zeta}{\partial c}\Big)\\ = & \ (L+G)\frac{\partial\eta}{\partial b} + (R-G)(v - \frac{\partial\eta}{\partial b})\\ = & \ (3R+G-R+G)\frac{\partial\eta}{\partial b} + (R-G)v\\ = & \ 2(R+G)\frac{\partial\eta}{\partial b} + (R-G)v\end{aligned}$$

By the same way,

$$\frac{C}{\Delta} = 2(R+G)\frac{\partial\zeta}{\partial c} + (R-G)v, \quad \frac{D}{\Delta} = (R+G)\Big(\frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}\Big), \quad \frac{E}{\Delta} = (R+G)\Big(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\Big), \quad \frac{F}{\Delta} = (R+G)\Big(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\Big)$$

For convenience's sake, in the paticular case, for $(41)_C$ and $(45)_C$ to hold, it is sufficient to be as follows :

$$(59)_C \quad (R+G)\Delta \equiv \frac{1}{2}k, \quad (R-G)\Delta \equiv K$$

For the equations $(56)_C$ and $(57)_C$,

$$(60)_C \quad \Rightarrow \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial\xi}{\partial a} + Kv & \frac{1}{2}k\Big(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\Big) & \frac{1}{2}k\Big(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\Big) \\ \frac{1}{2}k\Big(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\Big) & k\frac{\partial\eta}{\partial b} + Kv & \frac{1}{2}k\Big(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\Big) \\ \frac{1}{2}k\Big(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\Big) & \frac{1}{2}k\Big(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\Big) & k\frac{\partial\zeta}{\partial c} + Kv \end{bmatrix}$$

If, moreover, the condition $(54)_C$: $R = -G$ holds, $k = 0$ holds , and the followings hold :

$$(61)_C \quad A = B = C = Kv, \quad D = E = F = 0.$$

3.3.3. **Consideration of Elastic Fluid by Cauchy**.

We show the equation number of fluid by Cauchy in below, with $(\cdot)_{C^*}$ instead by $(\cdot)_C$ for discrimination with the equations of elastic in above.

• ¶ 17. Assumption of elastic fluid.

As the equations in equiblium :

$$(67)_{C^*} \quad \begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = 0, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = 0, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = 0, \end{cases}$$

and as the equations in motion :

$$(68)_{C^*} \quad \begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$$

Si de plus les valeurs de $G,\ H,\ I,\ L,\ M,\ N,\ P,\ Q,\ R$ deviennent indépecdantes en chaque point des directions assignées aux des $x,\ y$ et z, les conditions $(41)_C$ et $(45)_C$ seront vérifiées, et, en supposant la quantité v déterminée par l'equation $(47)_C$, ou, ce qui revient au même, par la suivante

$$(69)_{C^*} \quad v = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} = \nabla\cdot\mathbf{u} = div\ \mathbf{u}, \quad \mathbf{u} = (\xi,\ \eta,\ \zeta).$$

As the equilibrium of fluid :

$$(70)_{C^*} \quad \begin{cases} (R+G)\Big(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\Big) + 2R\frac{\partial v}{\partial x} + X = 0, \\ (R+G)\Big(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial z^2}\Big) + 2R\frac{\partial v}{\partial y} + Y = 0, \\ (R+G)\Big(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + \frac{\partial^2\zeta}{\partial z^2}\Big) + 2R\frac{\partial v}{\partial z} + Z = 0, \end{cases}$$

and as the equations in motion :

$$(71)_{C^*}\quad \begin{cases} (R+G)\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)+2R\frac{\partial v}{\partial x}+X=\frac{\partial^2\xi}{\partial t^2},\\ (R+G)\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}\right)+2R\frac{\partial v}{\partial y}+Y=\frac{\partial^2\eta}{\partial t^2},\\ (R+G)\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}\right)+2R\frac{\partial v}{\partial z}+Z=\frac{\partial^2\zeta}{\partial t^2},\end{cases}$$

By $(54)_C$

$$(72)_{C^*}\quad 2R\frac{\partial v}{\partial x}+X=0,\quad 2R\frac{\partial v}{\partial y}+Y=0,\quad 2R\frac{\partial v}{\partial z}+Z=0$$

$$(73)_{C^*}\quad 2R\frac{\partial v}{\partial x}+X=\frac{\partial^2\xi}{\partial t^2},\quad 2R\frac{\partial v}{\partial y}+Y=\frac{\partial^2\eta}{\partial t^2},\quad 2R\frac{\partial v}{\partial z}+Z=\frac{\partial^2\zeta}{\partial t^2}$$

On doit observer

- que la quantité v, déterminée par formule $(69)_{C^*}$, représente la dilatation qu'éprouve un volume très petet, mais choisi de manière à renfermer avec la molécule **m** un grand nombre de molécules voisines, tandis que ces molécules changent de position dans l'espace.
- Ajoutons que les formules $(72)_{C^*}$ et $(73)_{C^*}$, étant semblables aux formules $(63)_C$, $(72)_C$ et $(77)_C$ des pages 173, 175 and 176,[26] paraissent convenir à un système de molécules qui seraient disposées de manière à constituer un fluide élastique.

[6, p.248]

• ¶ 18. Verification of equations in elastic fluid.
By replacing $(a,\ b,\ c)$ of $(56)_C$ and $(57)_C$ with $(x,\ y,\ z)$, we get $(74)_{C^*}$, $(75)_{C^*}$ of the equivalence of $(56)_C$ and $(57)_C$.

• ¶ 19.

$$(67)_C \Rightarrow (76)_{C^*}\quad \begin{cases} \frac{\partial A}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial E}{\partial z}+X\Delta=0,\\ \frac{\partial F}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial D}{\partial z}+Y\Delta=0,\\ \frac{\partial E}{\partial x}+\frac{\partial D}{\partial y}+\frac{\partial C}{\partial z}+Z\Delta=0,\end{cases} \Rightarrow \begin{bmatrix} A & F & E\\ F & B & D\\ E & D & C\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}+\Delta\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}=\mathbf{0}$$

• ¶ 20.

We get the tensor from $(76)_{C^*}$ as follows :

$$(77)_{C^*}\quad \begin{bmatrix} A & F & E\\ F & B & D\\ E & D & C\end{bmatrix}$$

We reduce $(74)_C$, $(75)_C$ into as follows :

$$(60)_C \Rightarrow (78)_{C^*}\quad \begin{bmatrix} A & F & E\\ F & B & D\\ E & D & C\end{bmatrix}=\begin{bmatrix} k\frac{\partial\xi}{\partial x}+Kv & \frac{1}{2}k\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right) & \frac{1}{2}k\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right)\\ \frac{1}{2}k\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right) & k\frac{\partial\eta}{\partial y}+Kv & \frac{1}{2}k\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right)\\ \frac{1}{2}k\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right) & \frac{1}{2}k\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right) & k\frac{\partial\zeta}{\partial z}+Kv\end{bmatrix}$$

By replacing $R+G$ and $2R$ of $(70)_C$ and $(71)_C$ with followings :

$$C_1\equiv R+G=\frac{k}{2\Delta},\quad C_2\equiv 2R=\frac{k+2K}{2\Delta}$$

[26] Equations $(63)_C$, $(72)_C$ and $(77)_C$ of p.173, 175, 176 are included in [5], which are as follows :

$$(63)_C\quad \frac{\partial l(p)}{\partial x}=k(X-\frac{\partial u}{\partial t}),\quad \frac{\partial l(p)}{\partial y}=k(Y-\frac{\partial v}{\partial t}),\quad \frac{\partial l(p)}{\partial z}=k(Z-\frac{\partial w}{\partial t});$$

$$(72)_C\quad \frac{\partial l(\mathcal{P})}{\partial x}=kX,\quad \frac{\partial l(\mathcal{P})}{\partial y}=kY,\quad \frac{\partial l(\mathcal{P})}{\partial z}=kZ;$$

$$(77)_C\quad k\frac{\partial u}{\partial t}=\frac{\partial\infty}{\partial x},\quad k\frac{\partial v}{\partial t}=\frac{\partial\infty}{\partial y},\quad k\frac{\partial w}{\partial t}=\frac{\partial\infty}{\partial z};$$

As the equations in equilibrium of fluid :

$$(79)_{C^*}\quad \begin{cases} C_1\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)+C_2\frac{\partial v}{\partial x}+X=0,\\ C_1\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}\right)+C_2\frac{\partial v}{\partial y}+Y=0,\\ C_1\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}\right)+C_2\frac{\partial v}{\partial z}+Z=0,\end{cases}$$

and as the equations in motion of fluid :

$$(80)_{C^*}\quad \begin{cases} C_1\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)+C_2\frac{\partial v}{\partial x}+X=\frac{\partial^2\xi}{\partial t^2},\\ C_1\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}\right)+C_2\frac{\partial v}{\partial y}+Y=\frac{\partial^2\eta}{\partial t^2},\\ C_1\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}\right)+C_2\frac{\partial v}{\partial z}+Z=\frac{\partial^2\zeta}{\partial t^2}.\end{cases}$$

• ¶ 21. Comparison with Navier's equation in elasticity.

Cauchy says : for the reduction of the equations $(79)_{C^*}$ and $(80)_{C^*}$ to Navier's equations([26]) to determine the law of equilibrium and elasticity, it is necessary to assume such as the condition which we have mentioned above :

$$(81)_{C^*}\quad k=2K$$

• ¶ 22. Summaries on Navier's equations in elasticity.

> On voir au rest que, si l'on considère un corps élastique comme un système de points matériels qui agissent les uns sur les autres à de très petites distances, les lois de l'equilibre ou du mouvement intérieur de ce corps seront exprimées dans beaucoup de cas par des équations différentes de celles qu'a données M.Navier.
>
> - Les formules $(67)_{C^*}$ et $(68)_{C^*}$ paraissent spécialement applicables au cas où, l'élasticité n'étant pas la même dans les diverses directions, le corps offre trois axes d'élasticité rectangulaires entre eux, et parallèles aux axes des x, des y et des z.
> - Les formules $(70)_{C^*}$ et $(71)_{C^*}$, au contraire, semblent devoire s'appliquer au cas où le corps est également élastique dans tous les sens ; et alors on retrouvera les formules de M.Navier, si l'on attribue à la quantité G une valeur nulle.
> - Ajoutons que, si, dans les formules $(67)_{C^*}$ et $(68)_{C^*}$, on réduit à zéro, non seulement la quantité G, mais encore les quantités de même espèce H et I, ces formules deviendront respectivement [6, pp.251-252]

If $G=0$ then we get as the equations of equiblium in equal elasticity :

$$(67)_{C^*}\quad\Rightarrow\quad(83)_{C^*}\quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2}+R\frac{\partial^2\xi}{\partial y^2}+Q\frac{\partial^2\xi}{\partial z^2}+2R\frac{\partial^2\eta}{\partial x\partial y}+2Q\frac{\partial^2\zeta}{\partial z\partial x}+X=0,\\ R\frac{\partial^2\eta}{\partial x^2}+M\frac{\partial^2\eta}{\partial y^2}+P\frac{\partial^2\eta}{\partial z^2}+2P\frac{\partial^2\zeta}{\partial y\partial z}+2R\frac{\partial^2\xi}{\partial x\partial y}+Y=0,\\ Q\frac{\partial^2\zeta}{\partial x^2}+P\frac{\partial^2\zeta}{\partial y^2}+N\frac{\partial^2\zeta}{\partial z^2}+2Q\frac{\partial^2\xi}{\partial z\partial x}+2P\frac{\partial^2\eta}{\partial y\partial z}+Z=0,\end{cases}$$

and as the equations of motion in equal elasticity:

$$(68)_{C^*}\quad\Rightarrow\quad(84)_{C^*}\quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2}+R\frac{\partial^2\xi}{\partial y^2}+Q\frac{\partial^2\xi}{\partial z^2}+2R\frac{\partial^2\eta}{\partial x\partial y}+2Q\frac{\partial^2\zeta}{\partial z\partial x}+X=\frac{\partial^2\xi}{\partial t^2},\\ R\frac{\partial^2\eta}{\partial x^2}+M\frac{\partial^2\eta}{\partial y^2}+P\frac{\partial^2\eta}{\partial z^2}+2P\frac{\partial^2\zeta}{\partial y\partial z}+2R\frac{\partial^2\xi}{\partial x\partial y}+Y=\frac{\partial^2\eta}{\partial t^2},\\ Q\frac{\partial^2\zeta}{\partial x^2}+P\frac{\partial^2\zeta}{\partial y^2}+N\frac{\partial^2\zeta}{\partial z^2}+2Q\frac{\partial^2\xi}{\partial z\partial x}+2P\frac{\partial^2\eta}{\partial y\partial z}+Z=\frac{\partial^2\zeta}{\partial t^2}\end{cases}$$

(cf. We discuss the concurrence in §4.2, §4.3 and §4.5)

3.4. Saint-Venant's works and his tensor.

3.4.1. Evaluation of Saint-Venant's works.

In the bibliographies on Saint-Venant's works,[27] his main theories on the elastic or fluid mechanics exist in

- only four pages of the extract [52],
- the report [7] of its papers recognized by one of the then judges of MAS : Cauchy,
- his commentary on Clebsch [8] and
- his death notice [4] by J.Boussinesq and A.Flamant, and so on.

[27] Adhémar Jean Claude Barré de Saint-Venant (1797-1886). J.Boussinesq and A.Flamant list over 137 Saint-Venant's works in [4].

J.J.O'Connor and E.F.Robertson [32] say Saint-Venant's works are under-estimated up to now. We see it because

- Saint-Venant presents [52] to *Académie des Sciences* in 1843 and publishes in 1845. Stokes presents [55] in 1845 and publishes in 1849,
- both tensors are similar and expression by t_{ij} is equal.
- Cauchy [7] evaluates Saint-Venant's [52] and expresses the report [7] from the viewpoint of a then judge of the same MAS : *Mémoires des Académie des Sciences*, in which [7] is the next paper located after [52].

3.4.2. **Saint-Venant's tensor.**

Here we deal only with [52]. Saint-Venant explains the object of his paper [52] to simplfy the description and calculation of molecular relation without setting the molecular function. His method is an epoc-making method of tensor :

> Cette Note a pour objet de faciliter l'examen du Mémoire de 1834 et de ce qui y a été ajouté en 1837, en simplifiant, comme on va le dire, l'exposition du point principal, qui est la recherche des formules des pressions dans l'intérieur des fluides en mouvement, sans faire de supposition sur la grandeur des atractions et répulsions des molécules en founction, soit de leurs distances, soit de leurs vitesses relatives. [52, p.1240]

We show Saint-Venant's tensor, which seems to hint Stokes, from the extract [52]. ξ, η, ζ : velocities on the arbitrary point m of a fluid in motion of paralleled direction of the coordinate x, y, z respectively. P_{xx}, P_{yy}, P_{zz} : normal pressure and P_{yz}, P_{zx}, P_{xy} : tangential pressure with double sub-indices showing perpendicular plane and direction of decomposition, if strictly speaking, such as follows :

> P_{xx}, P_{yy}, P_{zz} les *pressions normales* supportées au même point par l'unité superficielle de petites faces perpendiculaires aux x, aux y, aux z, c'est-à-dire les composantes, dans un sens normale à ces faces fictives, des pressions qui s'exercent à travers ;
>
> P_{yz}, P_{zx}, P_{xy} les *pressions tangentielles* sur les mêmes faces et dans les trios sens, c'est-à-dire les composantes, parallèlment aux faces, des pressions dont nous venons de parler ;
>
> - la premiàre sous-lettre désignant toujours la face. par la coordonnée qui lui est perpendiculaire, et
> - la deuxième spécifiant le sens de la décomposition. [52, p.1240]

$$(1)_{SV} \quad \frac{P_{xx}-P_{yy}}{2\left(\frac{d\xi}{dx}-\frac{d\eta}{dy}\right)} = \frac{P_{zz}-P_{xx}}{2\left(\frac{d\zeta}{dx}-\frac{d\eta}{dz}\right)} = \frac{P_{yy}-P_{zz}}{2\left(\frac{d\eta}{dy}-\frac{d\zeta}{dz}\right)} = \frac{P_{yz}}{\frac{d\eta}{dz}+\frac{d\zeta}{dy}} = \frac{P_{zx}}{\frac{d\zeta}{dx}+\frac{d\xi}{dz}} = \frac{P_{xy}}{\frac{d\xi}{dy}+\frac{d\eta}{dx}} = \varepsilon,$$

where, we put

$$\frac{1}{3}\left(P_{xx}+P_{yy}+P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right) = \pi,$$

We put normal pressure respectively as follows :

$$(2)_{SV} \quad P_{xx} = \pi + 2\varepsilon\frac{d\xi}{dx}, \quad P_{yy} = \pi + 2\varepsilon\frac{d\eta}{dy}, \quad P_{zz} = \pi + 2\varepsilon\frac{d\zeta}{dz},$$

From $(1)_{SV}$, we get tangential pressure respectively as follows :

$$(3)_{SV} \quad P_{yz} = \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right), \quad P_{zx} = \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right), \quad P_{xy} = \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right).$$

From $(2)_{SV}$, we get π as follows :

$$P_{xx}+P_{yy}+P_{zz} = 3\pi + 2\varepsilon\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right) \quad \Rightarrow \quad \pi = \frac{1}{3}\left(P_{xx}+P_{yy}+P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right).$$

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi + 2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \pi + 2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) & \pi + 2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}, \quad \text{where} \quad \pi = \frac{1}{3}\left(P_{xx} + P_{yy} + P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right). \tag{94}$$

Saint-Venant says by using his theory, we can deduce the concurrence with Navier, Cauchy and Poisson :

> Si l'on remplace π par $\varpi - \varepsilon(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz})$, et si l'on substitue les équations $(2)_{SV}$ et $(3)_{SV}$ dans les relations connues entre les pressions et les forces accélératrices, on obtient, en supposant ε le même en tous les points du fluide, les équations différentielles données le 18 mars 1822 par M.Navier (*Mémoires de l'Institut*, t.VI), en 1828 par M.Cauchy (*Exercices de Mathématiques*, p.187)[28], et le 12 octobre 1829 par M.Poisson (même *Mémoire*, p.152)[29].
>
> La quantité variable ϖ ou π n'est autre chose, dans les liquides, que la *pression normale moyenne* en chaque point. [52, p.1243]

This paper[52] seems to give Stokes a hint of tensor (101), because we can see by comparing[30] t_{ij} with Stokes' t_{ij} (102) :

$$\begin{aligned} t_{ij} &= \left(\pi + 2\varepsilon v_{i,j} - \gamma\right)\delta_{ij} + \gamma \\ &= \left(\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) + 2\varepsilon v_{i,j} - \gamma\right)\delta_{ij} + \gamma \\ &= (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}) \quad \Leftarrow \quad 2\varepsilon v_{i,j}\delta_{ij} = \varepsilon(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij} \end{aligned} \tag{95}$$

$$\text{where} \quad \gamma = \varepsilon(v_{i,j} + v_{j,i}), \quad v_{k,k} = \sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \cdots \text{Einstein's convention}$$

Here, using (95), if we put[31] $P_{xx} = P_{yy} = P_{zz} = -p$ by Stokes principle in § 3.5, then (95) is equivalent to Stokes' t_{ij} as follows :

$$\begin{aligned} t_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}) &= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}) \\ \Rightarrow \quad \text{Stokes'} \ - t_{ij} &= (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i}) \Rightarrow (102). \end{aligned}$$

Moreover Saint-Venant assume that : if we put $\pi = \varpi - \varepsilon\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = \varpi - \varepsilon v_{k,k}$ then

$$t_{ij} = \left(\varpi - \varepsilon v_{k,k} + 2\varepsilon v_{i,j} - \gamma\right)\delta_{ij} + \gamma = (\varpi - \varepsilon v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}) \tag{96}$$

---

[28] Cauchy [5, p.226]

[29] Poisson [47, p.152] $(7\text{-}9)_{Pf}$=(80).

[30] In our paper, we cite the description of t_{ij} of the tensor : of Poisson and Cauchy, from C.Truesdell[57], of Navier, from G.Darrigol [12]. in else case by ourself or Schlichting[54].

[31] cf.I.Imai [17, p.185].

TABLE 12. Concurrence of tensor from the viewpoint of Saint-Venant with others

if we put $\pi = \varpi - \varepsilon\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = \varpi - \varepsilon v_{k,k}$ then Saint-Venant's (96) $t_{ij}^{SV} = \varpi\delta_{ij} - \varepsilon_1 v_{k,k}\delta_{ij} + \varepsilon_2(v_{i,j} + v_{j,i})$	$\Rightarrow \quad \varpi = 0$	$\Rightarrow \quad \varpi = -p$
$\varepsilon_1 = \varepsilon, \quad \varepsilon_2 = -\varepsilon$	Navier's (49) $t_{ij}^{N^e} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$	
$\varepsilon_1 = -\lambda, \quad \varepsilon_2 = \mu$	Cauchy's $t_{ij}^{C^e} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$	Poisson's $t_{ij}^{P^e} = (-p + \lambda v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
$\varepsilon_1 = \frac{2}{3}\mu, \quad \varepsilon_2 = \mu$		Stokes' (102) $t_{ij}^{S^f} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$

By the way, we check the coincidence Saint-Venant's tensor with Stokes'(101) only $(1,1)$ element or P_1.

$$
\begin{aligned}
P_1 \text{ of } (94) \quad &\Rightarrow \quad \pi + 2\varepsilon\frac{d\xi}{dx} \\
&= \quad -p + (2 - \frac{2}{3})\varepsilon\frac{d\xi}{dx} - \frac{2\varepsilon}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) \\
&= \quad -p + \varepsilon\Big\{\frac{4}{3}\frac{d\xi}{dx} - \frac{2}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} \\
&= \quad -p + 2\varepsilon\Big\{\frac{2}{3}\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} \\
&= \quad -p + 2\varepsilon\Big\{\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} \\
&= \quad -p + 2\varepsilon\Big(\frac{d\xi}{dx} - \delta\Big) \quad \Rightarrow \quad p - 2\mu\Big(\frac{du}{dx} - \delta\Big) \quad \Rightarrow \quad P_1 \text{ of } (101).
\end{aligned}
$$

where,

$$\pi = \frac{1}{3}\Big(P_{xx} + P_{yy} + P_{zz}\Big) - \frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) \equiv -p - 2\varepsilon\delta, \quad \delta = \frac{1}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big).$$

Else elements are coincident with (101) in the same way.

From here, we get Poisson, Navier and Cauchy's t_{ij} as follows :

- $t_{ij} = \varpi\delta_{ij} - \varepsilon_1 v_{k,k}\delta_{ij} + \varepsilon_2(v_{i,j} + v_{j,i})$, where, $\varpi = -p, \ -\varepsilon_1 = \lambda, \ \varepsilon_2 = \mu$
  $\Rightarrow$ Poisson's $t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,

- $t_{ij} = \varpi\delta_{ij} - \varepsilon_1 v_{k,k}\delta_{ij} + \varepsilon_2(v_{i,j} + v_{j,i})$, where, $\varpi = 0, \ -\varepsilon_1 = \lambda, \ \varepsilon_2 = \mu$
  $\Rightarrow$ Cauchy's $t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,

- $t_{ij} = \varpi\delta_{ij} - \varepsilon(v_{k,k}\delta_{ij} + v_{i,j} + v_{j,i})$, where, $\varpi = 0$,
  $\Rightarrow$ Navier's $t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$,

  Moreover, we can add Stokes'

- $t_{ij} = \varpi\delta_{ij} - \varepsilon_1 v_{k,k}\delta_{ij} + \varepsilon_2(v_{i,j} + v_{j,i})$, where, $\varpi = -p, \ \varepsilon_1 = \frac{2}{3}\mu, \ \varepsilon_2 = \mu$
  $\Rightarrow$ Stokes'$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$.

3.5. Stokes' principle, equations and tensor.

Stokes says in [55, p.80] :[32]

> If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle :

[32]Stokes [55, pp.78-105]Section 1. Explanation of the Theory of Fluid Motion proposed. Formulation of the Differential Equations. Application of these Equations to a few simple cases.

- That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P ; and
- that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.

Stokes comments on Navier's equation :

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389)[33], but his principles differ from mine still more than do Poisson's. [55, p.77, footnote]

$$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0. \end{cases} \tag{97}$$

where Stokes says the coincidence with Poisson :

$$\varpi = p+\frac{\alpha}{3}(K+k)\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) \Rightarrow \nabla\varpi = \nabla p+\frac{\beta}{3}\nabla\cdot(\nabla\cdot\mathbf{u}). \tag{98}$$

> Observing that $\alpha(K+k)\equiv\beta$, this value of ϖ reduces Poisson's equation $(9)_{Pf}$ (=(80) in our renumbering) to the equation $(12)_S$ of this paper.

By the way, $(12)_S$ turns to :

$$\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\Big(\frac{4}{3}\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+\frac{1}{3}\frac{d^2v}{dxdy}+\frac{1}{3}\frac{d^2w}{dxdz}\Big)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\Big(\frac{d^2v}{dx^2}+\frac{4}{3}\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+\frac{1}{3}\frac{d^2u}{dxdy}+\frac{1}{3}\frac{d^2w}{dydz}\Big)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{4}{3}\frac{d^2w}{dz^2}+\frac{1}{3}\frac{d^2u}{dxdz}+\frac{1}{3}\frac{d^2w}{dydz}\Big)=0. \end{cases}$$

or

$$\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\frac{\mu}{3}\Big(4\frac{d^2u}{dx^2}+3\frac{d^2u}{dy^2}+3\frac{d^2u}{dz^2}+\frac{d^2v}{dxdy}+\frac{d^2w}{dxdz}\Big)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\frac{\mu}{3}\Big(3\frac{d^2v}{dx^2}+4\frac{d^2v}{dy^2}+3\frac{d^2v}{dz^2}+\frac{d^2u}{dxdy}+\frac{d^2w}{dydz}\Big)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\frac{\mu}{3}\Big(3\frac{d^2w}{dx^2}+3\frac{d^2w}{dy^2}+4\frac{d^2w}{dz^2}+\frac{d^2u}{dxdz}+\frac{d^2w}{dydz}\Big)=0, \end{cases} \tag{99}$$

where when we use vectoriel notation after replacing with $\mathbf{f}\equiv(X,Y,Z)$, we get :

$$\rho(\frac{D\mathbf{u}}{Dt}-\mathbf{f})+\nabla p-\mu\Big(\Delta\mathbf{u}+\frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\Big)=0 \quad \text{or} \quad \frac{D\mathbf{u}}{Dt}-\frac{\mu}{\rho}\Delta\mathbf{u}-\frac{1}{3\rho}\nabla(\nabla\cdot\mathbf{u})+\frac{1}{\rho}\nabla p=\mathbf{f}$$

Stokes proposes the Stokes' approximate equations in [55, p.93] :

$$(13)_S \quad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2})=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2})=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2})=0, \end{cases} \qquad \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0. \tag{100}$$

Stokes proposes that :

> These equations are applicable to the determination of the motion of water in pipes and canala, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

Here we shall trace his deduction with Stokes' tensor :

---

[33] Navier[27].

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p-2\mu\Big(\frac{du}{dx}-\delta\Big) & -\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & -\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) \\ -\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & p-2\mu\Big(\frac{dv}{dy}-\delta\Big) & -\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) \\ -\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) & -\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & p-2\mu\Big(\frac{dw}{dz}-\delta\Big) \end{bmatrix}, \quad \text{where} \quad 3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \tag{101}$$

Here, he reads, "it may also be very easily provided directly that the value of 3δ, the rate of cubical dilation". By the way, Stokes' tensor is described compactly as follows :

$$\begin{aligned} -t_{ij} &= \{p-2\mu(v_{i,j}-\delta)+\gamma\}\delta_{ij}-\gamma \\ &= \{p-2\mu v_{i,j}\}\delta_{ij}+\gamma(-\delta_{ij}+\delta_{ij}-1) \quad \Leftarrow \quad 2\mu\delta\delta_{ij}=\mu(v_{i,j}+v_{j,i})\delta_{ij}=\gamma\delta_{ij} \\ &= (p+2\mu\gamma)\delta_{ij}-\gamma \\ &= (p+\frac{2}{3}\mu v_{k,k})\delta_{ij}-\mu(v_{i,j}+v_{j,i}), \end{aligned} \tag{102}$$

Here, the sign of $-t_{ij}$ depends on the location of the tensor in the equation, and we consider the coincident with (97). [34] We see Stokes' tensor comes from Saint-Venant's tensor. From here, the article by J.J.O'Connor and E.F.Robertson point out this resemblance.[35]

By D'Alembert's principle

$$\begin{cases} \rho\Big(\frac{Du}{Dt}-X\Big)+\frac{dP_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}=\rho\Big(\frac{Du}{Dt}-X\Big)+P=0, \\ \rho\Big(\frac{Dv}{Dt}-Y\Big)+\frac{dP_2}{dy}+\frac{dT_3}{dx}+\frac{dT_1}{dz}=\rho\Big(\frac{Dv}{Dt}-Y\Big)+Q=0, \\ \rho\Big(\frac{Dw}{Dt}-Z\Big)+\frac{dP_3}{dz}+\frac{dT_2}{dx}+\frac{dT_1}{dy}=\rho\Big(\frac{Dw}{Dt}-Z\Big)+R=0 \end{cases} \tag{103}$$

By (101) and (103), we get (100). We see in fact as follows : we seek the t_{ij} such that :

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix}$$

Using (101) and (103),

$$\begin{cases} \frac{d}{dx}\Big\{p-2\mu\Big(\frac{du}{dx}\Big)+2\mu\frac{1}{3}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}+\frac{d}{dy}\Big\{-\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\Big\}+\frac{d}{dz}\Big\{-\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\Big\}, \\ \frac{d}{dy}\Big\{p-2\mu\Big(\frac{dv}{dy}\Big)+2\mu\frac{1}{3}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}+\frac{d}{dx}\Big\{-\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\Big\}+\frac{d}{dz}\Big\{-\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\Big\}, \\ \frac{d}{dz}\Big\{p-2\mu\Big(\frac{dw}{dz}\Big)+2\mu\frac{1}{3}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}+\frac{d}{dx}\Big\{-\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\Big\}+\frac{d}{dy}\Big\{-\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\Big\} \end{cases}$$

$$= \begin{cases} \frac{d}{dx}p-\mu\Big\{\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)+\frac{1}{3}\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}, \\ \frac{d}{dy}p-\mu\Big\{\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)+\frac{1}{3}\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}, \\ \frac{d}{dz}p-\mu\Big\{\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)+\frac{1}{3}\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}, \end{cases} = \begin{cases} \frac{d}{dx}p-\mu\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big), \\ \frac{d}{dy}p-\mu\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big), \\ \frac{d}{dz}p-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big) \end{cases}$$

Therefore we get (100).

By the modern vectoriel expression, if we take $\mathbf{f}=(X,Y,Z)$, $\nu\equiv\frac{\mu}{\rho}$, and as Stokes says that we may put $Du/Dt=\partial\mathbf{u}/\partial t$, then (100) turns out as follows :

$$\frac{\partial\mathbf{u}}{\partial t}-\nu\Delta\mathbf{u}+\frac{1}{\rho}\nabla p=\mathbf{f}, \quad \text{div } \mathbf{u}=0.$$

[34] Schlichting writes Stokes' tensor with the minus sign as follows :

$$\sigma_{ij}=-p\delta_{ij}+\mu\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big)-\frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$$

[54, p.58, in footnote]

[35] cf. J.J.O'Connor, E.F.Robertson,→ http://www-groups.dcs.st-and.ac.uk/ history/Printonly/Saint-Venant.html.[32]

By the way, here we shall get the tensor of Stokes equations from Navier's (2). We put as the same as Stokes equations :

$$\begin{cases} \rho\left(\frac{du}{dt}-X\right)+\frac{dp}{dx}-\varepsilon\left(3\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+2\frac{d^2v}{dxdy}+2\frac{d^2w}{dxdz}\right)+\frac{du}{dx}\cdot u+\frac{du}{dy}\cdot v+\frac{du}{dz}\cdot w=0\ ; \\ \rho\left(\frac{dv}{dt}-Y\right)+\frac{dp}{dy}-\varepsilon\left(\frac{d^2v}{dx^2}+3\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+2\frac{d^2u}{dxdy}+2\frac{d^2w}{dydz}\right)+\frac{dv}{dx}\cdot u+\frac{dv}{dy}\cdot v+\frac{dv}{dz}\cdot w=0\ ; \\ \rho\left(\frac{dw}{dt}-Z\right)+\frac{dp}{dz}-\varepsilon\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+3\frac{d^2w}{dz^2}+2\frac{d^2u}{dxdz}+2\frac{d^2v}{dydz}\right)+\frac{dw}{dx}\cdot u+\frac{dw}{dy}\cdot v+\frac{dw}{dz}\cdot w=0\ ; \end{cases}$$

Using D'Alembert principle (103), we transform the terms of the coefficient of 3 with $3=2+1$ and the last 2 terms of the coefficient of 2 with $2=1+1$, respectively. We show here the viscosity term as follows :

$$\begin{cases} -\varepsilon\left((2\frac{d^2u}{dx^2}+\frac{d^2u}{dx^2})+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+2\frac{d^2v}{dxdy}+2\frac{d^2w}{dxdz}\right); \\ -\varepsilon\left(\frac{d^2v}{dx^2}+(2\frac{d^2v}{dy^2}+\frac{d^2v}{dy^2})+\frac{d^2v}{dz^2}+2\frac{d^2u}{dxdy}+2\frac{d^2w}{dydz}\right); \\ -\varepsilon\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+(2\frac{d^2w}{dz^2}+\frac{d^2w}{dz^2})+2\frac{d^2u}{dxdz}+2\frac{d^2v}{dydz}\right); \end{cases}$$

$$= \begin{cases} -\varepsilon\left\{2\frac{d^2u}{dx^2}+\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}+\frac{d}{dy}\left(\frac{du}{dy}+\frac{dv}{dx}\right)+\frac{d}{dz}\left(\frac{du}{dz}+\frac{dw}{dx}\right)\Big\}; \\ -\varepsilon\left\{\frac{d}{dx}\left(\frac{du}{dy}+\frac{dv}{dx}\right)+\left\{2\frac{d^2v}{dy^2}+\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}+\frac{d}{dz}\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\right\}; \\ -\varepsilon\left\{\frac{d}{dx}\left(\frac{dw}{dx}+\frac{du}{dz}\right)+\frac{d}{dy}\left(\frac{dv}{dz}+\frac{dw}{dy}\right)+\left\{2\frac{d^2w}{dz^2}+\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}\right\}; \end{cases}$$

We get the tensor t_{ij} :

$$\begin{bmatrix} p-\varepsilon\left(2\frac{du}{dx}+\delta\right) & -\varepsilon\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\varepsilon\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\varepsilon\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-\varepsilon\left(2\frac{dv}{dy}+\delta\right) & -\varepsilon\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\varepsilon\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\varepsilon\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-\varepsilon\left(2\frac{dw}{dz}+\delta\right) \end{bmatrix}, \quad \text{where} \quad \delta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}, \qquad (104)$$

or

$$\begin{bmatrix} p-2\varepsilon\left(\frac{du}{dx}+\delta\right) & -\varepsilon\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\varepsilon\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\varepsilon\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\varepsilon\left(\frac{dv}{dy}+\delta\right) & -\varepsilon\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\varepsilon\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\varepsilon\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\varepsilon\left(\frac{dw}{dz}+\delta\right) \end{bmatrix}, \quad \text{where} \quad 2\delta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}. \qquad (105)$$

Therefore we see (101), (104) and (105) are the equivalent invariant-tensors each other except for the sign of δ.

4. Concurrence

4.1. General concurrences.

We summarize some concurrence. We put C_T : the matrix consisted of the coefficients in the tensor matrix T, that are coefficients of derivatives of $\frac{\partial^2\xi}{\partial XX}$, $\frac{\partial^2\eta}{\partial XX}$, $\frac{\partial^2\zeta}{\partial XX}$ in respect to XX : subscriptions of C_{XX}. We define the coefficient matrix in elastics : C^e_T as follows :

$$C^e_T:\ \text{the coefficient of}\quad \begin{bmatrix} \frac{\partial^2u}{\partial x^2} & \frac{\partial^2u}{\partial y^2} & \frac{\partial^2u}{\partial z^2} & \frac{\partial^2v}{\partial x\partial y} & \frac{\partial^2w}{\partial z\partial x} \\ \frac{\partial^2v}{\partial x^2} & \frac{\partial^2v}{\partial y^2} & \frac{\partial^2v}{\partial z^2} & \frac{\partial^2w}{\partial y\partial z} & \frac{\partial^2u}{\partial x\partial y} \\ \frac{\partial^2w}{\partial x^2} & \frac{\partial^2w}{\partial y^2} & \frac{\partial^2w}{\partial z^2} & \frac{\partial^2u}{\partial z\partial x} & \frac{\partial^2v}{\partial y\partial z} \end{bmatrix},$$

then

$$(6\text{-}1)_{N^e} \quad\Rightarrow\quad C^e_T = -\varepsilon\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \quad\Rightarrow\quad (2)$$

C_C, C_N, C_P, C_{SV}, C_S : C_T by Cauchy, Navier, Poisson, Saint-Venan and Stokes, respectively.
C_1 : the same constant as in Table 1 or 2,

- In case of elastic solid ; if we assume $C_{N^e}=C_{P^e}=C_C=C_1C_T$, then :

$$\left[kC_{P^e}\cdots(6)_{P^e}\right] \iff \left[\varepsilon C_{N^e}\cdots(5\text{-}3)_{N^e}=(16)\right] \iff \left[RC_C\cdots(84)_{C^*}\right],$$

where, $C_{N^e}=C_{P^e}=C_C=C_1C_T$,

$$C_1=\begin{cases} k\equiv\frac{2\pi}{15}\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}=\frac{a^2}{3}\rho, \quad (57)\\ \varepsilon\equiv\frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \quad (3\text{-}9)_{N^e}, \quad (29)\\ R\equiv\frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr=\pm\frac{2\pi\Delta}{15}\int_0^\infty\left[r^4\mathrm{f}'(r)-r^3\mathrm{f}(r)\right]dr, \quad (52)_C, \quad (=(93)). \end{cases}$$

If we put $l\equiv\rho=r$:

$$\sum\frac{l^5}{\alpha^5}\frac{d.\frac{1}{l}f_{P^e}}{dl}=\int_0^\infty dl\ l^4 f_N(l)=\int_0^\infty dl\ \Delta l^3 f_C(l)=\pm\int_0^\infty dl\Delta\left[l^4\mathrm{f_C}'(l)-l^3\mathrm{f_C}(l)\right]$$

- In case of fluid : if we assume $C_{N^e}=C_{N^f}=C_{P^e}=C_C=C_1C_T$, then

$$\left[(7\text{-}9)_{P^f}=(80)\right] \iff (101)\text{ of S} \Leftarrow (94)\text{ of SV}$$

$$\left[RC_C\cdots(84)_{C^*}\right] \iff \varepsilon C_{N^e} \iff (2)\text{ of }N^f$$

where, $C_{N^e}=C_{N^f}=C_{P^e}=C_C=C_1C_T$

$$C_1=\begin{cases} k\equiv\frac{1}{30\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr}=\frac{2\pi}{15}\sum\frac{r^3}{4\pi\varepsilon^3}\frac{d.\frac{1}{r}fr}{dr}, \quad (3\text{-}8)_{P^f}, \quad (75),\\ \varepsilon\equiv\frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \quad (3\text{-}9)_{N^e}, \quad (29)\\ R\equiv\frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr=\pm\frac{2\pi\Delta}{15}\int_0^\infty\left[r^4\mathrm{f}'(r)-r^3\mathrm{f}(r)\right]dr, \quad (52)_C, \quad (=(93)). \end{cases}$$

If we put $l\equiv\rho=r$:

$$\sum\frac{l^3}{4\pi\varepsilon^3}\frac{d.\frac{1}{l}fl}{dl}=\int_0^\infty dl\ l^4 f_N(l)=\int_0^\infty dl\ \Delta l^3 f_C(l)=\pm\int_0^\infty dl\Delta\left[l^4\mathrm{f_C}'(l)-l^3\mathrm{f_C}(l)\right]$$

- In common case :

If we take off the integral symbol from the equations of Navier and Cauchy :

$$l^4 f_N(l)=\Delta l^3 f_C(l)=\pm\Delta\left[l^4\mathrm{f_C}'(l)-l^3\mathrm{f_C}(l)\right]$$

Devide by l^3 of each hand-sides :

$$lf_N(l)=\Delta f_C(l)=\pm\Delta\left[l\mathrm{f_C}'(l)-\mathrm{f_C}(l)\right]$$

Here, Δ : dencity denouted by Cauchy defined with $(48)_C$.

4.2. **Equations by Cauchy.**

$$(41)_C \quad G=H=I, \quad L=M=N, \quad P=Q=R.$$

$$(45)_C \quad L=3R.$$

$$(46)_C \quad \begin{cases} X=(R+G)\left(\frac{\partial^2\xi}{\partial a^2}+\frac{\partial^2\xi}{\partial b^2}+\frac{\partial^2\xi}{\partial c^2}\right)+2R\frac{\partial v}{\partial a},\\ Y=(R+G)\left(\frac{\partial^2\eta}{\partial a^2}+\frac{\partial^2\eta}{\partial b^2}+\frac{\partial^2\eta}{\partial c^2}\right)+2R\frac{\partial v}{\partial b},\\ Z=(R+G)\left(\frac{\partial^2\zeta}{\partial a^2}+\frac{\partial^2\zeta}{\partial b^2}+\frac{\partial^2\zeta}{\partial c^2}\right)+2R\frac{\partial v}{\partial c}, \end{cases} \qquad (106)$$

$$\text{where} \quad (47)_C \quad v=\frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}.$$

If $G=0$ then we get as the equations by Cauchy in equiblium :

$$(83)_{C^{*}}\quad\begin{cases}L\frac{\partial^2\xi}{\partial x^2}+R\frac{\partial^2\xi}{\partial y^2}+Q\frac{\partial^2\xi}{\partial z^2}+2R\frac{\partial^2\eta}{\partial x\partial y}+2Q\frac{\partial^2\zeta}{\partial z\partial x}+X=0,\\ R\frac{\partial^2\eta}{\partial x^2}+M\frac{\partial^2\eta}{\partial y^2}+P\frac{\partial^2\eta}{\partial z^2}+2P\frac{\partial^2\zeta}{\partial y\partial z}+2R\frac{\partial^2\xi}{\partial x\partial y}+Y=0,\\ Q\frac{\partial^2\zeta}{\partial x^2}+P\frac{\partial^2\zeta}{\partial y^2}+N\frac{\partial^2\zeta}{\partial z^2}+2Q\frac{\partial^2\xi}{\partial z\partial x}+2P\frac{\partial^2\eta}{\partial y\partial z}+Z=0,\end{cases}\qquad(107)$$

and as the equations in motion :

$$(84)_{C^{*}}\quad\begin{cases}L\frac{\partial^2\xi}{\partial x^2}+R\frac{\partial^2\xi}{\partial y^2}+Q\frac{\partial^2\xi}{\partial z^2}+2R\frac{\partial^2\eta}{\partial x\partial y}+2Q\frac{\partial^2\zeta}{\partial z\partial x}+X=\frac{\partial^2\xi}{\partial t^2},\\ R\frac{\partial^2\eta}{\partial x^2}+M\frac{\partial^2\eta}{\partial y^2}+P\frac{\partial^2\eta}{\partial z^2}+2P\frac{\partial^2\zeta}{\partial y\partial z}+2R\frac{\partial^2\xi}{\partial x\partial y}+Y=\frac{\partial^2\eta}{\partial t^2},\\ Q\frac{\partial^2\zeta}{\partial x^2}+P\frac{\partial^2\zeta}{\partial y^2}+N\frac{\partial^2\zeta}{\partial z^2}+2Q\frac{\partial^2\xi}{\partial z\partial x}+2P\frac{\partial^2\eta}{\partial y\partial z}+Z=\frac{\partial^2\zeta}{\partial t^2}\end{cases}\qquad(108)$$

When we check the coefficients of both $(83)_{C^{*}}$ and $(84)_{C^{*}}$ by using $(41)_C$ and $(45)_C$, then we get the coefficient matrix as follows :

$$(84)_{C^{*}}\Rightarrow\quad C_C=\begin{bmatrix}L&R&Q&2R&2Q\\R&M&P&2P&2R\\Q&P&N&2Q&2P\end{bmatrix}=R\begin{bmatrix}3&1&1&2&2\\1&3&1&2&2\\1&1&3&2&2\end{bmatrix}\quad\Rightarrow\quad(5\text{-}3)_{N^e}=(16)$$

4.3. Concurrence by Cauchy with Poisson.

$$(6)_{Pe}\quad\begin{cases}X-\frac{d^2u}{dt^2}+a^2\left(\frac{d^2u}{dx^2}+\frac{2}{3}\frac{d^2v}{dydx}+\frac{2}{3}\frac{d^2w}{dzdx}+\frac{1}{3}\frac{d^2u}{dy^2}+\frac{1}{3}\frac{d^2u}{dz^2}\right)=0,\\ Y-\frac{d^2v}{dt^2}+a^2\left(\frac{d^2v}{dy^2}+\frac{2}{3}\frac{d^2u}{dxdy}+\frac{2}{3}\frac{d^2w}{dzdy}+\frac{1}{3}\frac{d^2v}{dx^2}+\frac{1}{3}\frac{d^2v}{dz^2}\right)=0,\\ Z-\frac{d^2w}{dt^2}+a^2\left(\frac{d^2w}{dz^2}+\frac{2}{3}\frac{d^2u}{dxdz}+\frac{2}{3}\frac{d^2v}{dydz}+\frac{1}{3}\frac{d^2w}{dx^2}+\frac{1}{3}\frac{d^2w}{dy^2}\right)=0,\end{cases}\qquad(109)$$

$$\Rightarrow\quad\begin{cases}X-\frac{d^2u}{dt^2}+\frac{a^2}{3}\left(3\frac{d^2u}{dx^2}+2\frac{d^2v}{dydx}+2\frac{d^2w}{dzdx}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)=0,\\ Y-\frac{d^2v}{dt^2}+\frac{a^2}{3}\left(3\frac{d^2v}{dy^2}+2\frac{d^2u}{dxdy}+2\frac{d^2w}{dzdy}+\frac{d^2v}{dx^2}+\frac{d^2v}{dz^2}\right)=0,\\ Z-\frac{d^2w}{dt^2}+\frac{a^2}{3}\left(3\frac{d^2w}{dz^2}+2\frac{d^2u}{dxdz}+2\frac{d^2v}{dydz}+\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right)=0,\end{cases}$$

$$\Rightarrow C_{Pe}=\;k\begin{bmatrix}3&1&1&2&2\\1&3&1&2&2\\1&1&3&2&2\end{bmatrix}\quad\Rightarrow\;(84)_{C^{*}}\;(=(108))$$

4.4. Concurrence by Stokes with Poisson.

We get the concurrence of $(106)(=(46)_C)$ with $(80)\;(=(7\text{-}9)_{Pf})$ and $(97)(=(12)_S)$.

$$(7\text{-}9)_{Pf}\quad\begin{cases}\rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}),\\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}),\\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2})\\ \text{where }\varpi=p+\frac{\alpha}{3}(K+k)(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}),\end{cases}$$

$$\Rightarrow\quad\begin{cases}\rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}+\alpha(K+k)\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)+\frac{1}{3}\alpha(K+k)\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}+\alpha(K+k)\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)+\frac{1}{3}\alpha(K+k)\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}+\alpha(K+k)\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)+\frac{1}{3}\alpha(K+k)\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,\end{cases}$$

$$\Rightarrow(12)_S\quad\begin{cases}\rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0.\end{cases}$$

4.5. Concurrence by Cauchy with Navier via Poisson.

These coefficients are equal to Navier's (21), which is deduced by Poisson from (61)=(62)(= $(6)_{Pe}$). Namely (108) is equal to (110).

$$(6\text{-}1)_{Ne} \quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases} \tag{110}$$

$$\Rightarrow \begin{cases} X - \frac{d^2u}{dt^2} + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dydx} + 2\frac{d^2w}{dzdx}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dzdy}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) = 0 \end{cases} \tag{111}$$

$$\begin{cases} X - \frac{d^2u}{dt^2} + \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + 2\varepsilon\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + 2\varepsilon\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + 2\varepsilon\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \end{cases} \tag{112}$$

(112) has the possibility of concurrence with Cauchy, if $G = 0$, such that :

$$(46)_C \quad \begin{cases} X = (R+G)\left(\frac{\partial^2\xi}{\partial a^2} + \frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\right) + 2R\frac{\partial\nu}{\partial a}, \\ Y = (R+G)\left(\frac{\partial^2\eta}{\partial a^2} + \frac{\partial^2\eta}{\partial b^2} + \frac{\partial^2\eta}{\partial c^2}\right) + 2R\frac{\partial\nu}{\partial b}, \\ Z = (R+G)\left(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2} + \frac{\partial^2\zeta}{\partial c^2}\right) + 2R\frac{\partial\nu}{\partial c}, \end{cases} \tag{113}$$

$$\text{where} \qquad (47)_C \quad \nu = \frac{\partial\xi}{\partial a} + \frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}.$$

4.6. Concurrence of tensor.

We show the correlation of concurrence in Table 6 and Table 13.

5. Summary

It is called that the two constants theory is the one now accepted for isotropic, linear elasticity.[36] We show that the tensors and equations are formulated in this frame. Namely, they need the followings for concurrence of the deduction of tensor or equations by Navier, Poisson, Cauchy, Saint-Venant and Stokes:

- for elastic body or fluid, baratropy, isotropy, homogeneity and linearity,
- for tensor, symmetry,
- for pressure, perpendicular,
- for mutual concurrence of the two constants theory, neglect of C_2, which we define in the section 2 or in Table 1 or Table 2.

In paticular, we would like to emphasize the work of Saint-Venant, whose work is an epock-making for taking the concurrence among these pioneers of Navier-Stokes equations and contributing to Stokes.

6. Acknoledgements

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References

[1] J.D.Anderson, *A history of aerodynamics and its impact on flying machines*, Cambridge, 1997.

[2] D.F.J.Arago, *Note du Rédacteur*, Annales de chimie et de physique, **39**(1829), 107-110. (This is following with Navier[28], 99-107).

[3] D.H.Arnold, *The mécanique physique of Siméon Denis Poisson : The evolution and isolation in France of his approach to physical theory (1800-1840) I,II,III,IV,V,VI,VII,VIII,IX,X*, Arch. Hist. Exact Sci. **28-3**(1983) I:243-266, II:267-287, III:289-297, IV:299-320, V:321-342, VI:343-367, **29-1**(1983) VII:37-51, VIII:53-72, IX: 73-94, **29-4**(1984) X:287-307.

[4] J.Boussinesq and A.Flamant, *Notice sur la vie et les travaux de Barré de Saint-Venant*, 1886, Paris.

[36] Darrigol [12, p.121].

TABLE 13. Concurrences and variations of tensors

1	name	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elastic	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(5\text{-}4)_{N^e}$ $-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) & \left(\frac{du}{dy} + \frac{dv}{dx}\right) & \left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ \left(\frac{du}{dy} + \frac{dv}{dx}\right) & \left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right) & \left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ \left(\frac{dw}{dx} + \frac{du}{dz}\right) & \left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon\begin{bmatrix} \epsilon + 2\frac{du}{dx} & \frac{du}{dy} + \frac{dv}{dx} & \frac{dw}{dx} + \frac{du}{dz} \\ \frac{du}{dy} + \frac{dv}{dx} & \epsilon + 2\frac{dv}{dy} & \frac{dv}{dz} + \frac{dw}{dy} \\ \frac{dw}{dx} + \frac{du}{dz} & \frac{dv}{dz} + \frac{dw}{dy} & \epsilon + 2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$	We define the coefficient matrix in elastics : C_T^e as follows : C_T^e : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 w}{\partial z \partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y \partial z} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z \partial x} & \frac{\partial^2 v}{\partial y \partial z} \end{bmatrix},$ then $(6\text{-}1)_{N^e} \Rightarrow C_T^e = -\varepsilon\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \Rightarrow (2)$
1-2	Navier fluid (only linear part of (2))	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ (2) $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy} + \frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx} + \frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy} + \frac{dv}{dx}) & \epsilon' - 2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz} + \frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx} + \frac{du}{dz}) & -\varepsilon(\frac{dv}{dz} + \frac{dw}{dy}) & \epsilon' - 2\varepsilon\frac{dw}{dz} \end{bmatrix},$ where $\epsilon' = p - \varepsilon(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz})$	Samely, we define the coefficient matrix in fluid : C_T^f , which contains p in (1,1)-, (2,2)- and (3,3)-element. $(2) \Rightarrow C_T^f = \begin{bmatrix} p - 3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p - 3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p - 3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2-1	Poisson elastic	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(6)_{P^e}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon + 2\frac{du}{dx} & \frac{du}{dy} + \frac{dv}{dx} & \frac{dw}{dx} + \frac{du}{dz} \\ \frac{du}{dy} + \frac{dv}{dx} & \epsilon + 2\frac{dv}{dy} & \frac{dv}{dz} + \frac{dw}{dy} \\ \frac{dw}{dx} + \frac{du}{dz} & \frac{dv}{dz} + \frac{dw}{dy} & \epsilon + 2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$	$(6)_{P^e}$ $\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = 0. \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = 0. \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = 0. \end{cases}$ $\Rightarrow C_T^e = -\frac{a^2}{3}\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
2-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & \pi + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \pi + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ \pi + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix},$ where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$	$(7\text{-}9)_{Pf} \Rightarrow C_T^f = \begin{bmatrix} \varpi + \beta & \beta & \beta & 0 & 0 \\ \beta & \varpi + \beta & \beta & 0 & 0 \\ \beta & \beta & \varpi + \beta & 0 & 0 \end{bmatrix}.$ According to Stokes : if we put $\varpi = p + \frac{\alpha}{3}(K + k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$ $\Rightarrow C_T^f = \begin{bmatrix} p + \frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p + \frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p + \frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix} \Rightarrow (12)_S\ (= (101)).$ Remark : $\alpha(K + k) = \beta.$
3	Cauchy system	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a} + K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b} + K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c} + K\nu \end{bmatrix},$ where $\nu = \frac{\partial\xi}{\partial a} + \frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}$	$(46)_C \Rightarrow C_T^e = \begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix}$ $\Rightarrow R\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix},$ where $P = Q = R,\quad L = M = N,\quad L = 3R.$
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $\begin{bmatrix} \pi + 2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \pi + 2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) & \pi + 2\varepsilon\frac{d\zeta}{dz} \end{bmatrix},$ where $\pi = \frac{1}{3}\left(P_{xx} + P_{yy} + P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)$ $\equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)$ (94)	non description in [52].
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$ tensor = −1 × $\begin{bmatrix} p - 2\mu\left(\frac{du}{dx} - \delta\right) & -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - 2\mu\left(\frac{dv}{dy} - \delta\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - 2\mu\left(\frac{dw}{dz} - \delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$ (101)	$(12)_S \Rightarrow C_T^f = \begin{bmatrix} -p + \frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p + \frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p + \frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix} \Rightarrow (101).$ Remark : $\frac{4}{3}\mu = 2\mu(1 - \frac{1}{3})$

[5] A.L.Cauchy, *Sur les équations qui expriment les conditions de l'équilibre ou les lois du mouvment intérieur d'un corps solide, élastique ou non élastique*, Exercises de Mathématique, **3**(1828); Œuvres complètes D'Augustin Cauchy, (Ser. 2) **8**(1890), 195-226.

[6] A.L.Cauchy, *Sur l'équilibre et le mouvement d'un système de points matériels sollicités par des forces d'attraction ou de répulsion mutuelle*, Exercises de Mathématique, **3**(1828); Œuvres complètes D'Augustin Cauchy (Ser. 2) **8**(1890), 227-252.

[7] A.L.Cauchy, *Rapport sur divers Mémoires de M. de Saint-Venant relatifs à la mécanique rationalle et à la mécanique appliquèe*, Académie des Sciences, *Comptes-rendus hebdomadaires des séances*, **17**(1843), 1234-1236. (cf. This follows after [52].)

[8] A.Clebsch, *Théorie de l'elasticité des corps solides - de Clebsch traduite par MM. B. de Saint-Venant et Flemant avec des Notes étendues de M. de Saint-Venant*, 1883. (cf. Préface de l'auteur : Carlsruhe, le 30 septembre 1862, first reprint by Johnson Reprint Co. : 1966).

[9] A.Cournot, *Mémoire sur les lois du mouvement des fluides ; par M.Navier. (Mém. de l'Acad. des Science ; Tom. VI, p. 389)*, Bulletin des sciences mathématiques, astromatiques, physiques et chimiques, **10**(1828), 11-14. (The title number : No.10.)

[10] G.Darboux, *Œuvres de Fourier. Publiées par les soins de M.Gaston Darboux*, Tome Premier, Paris, 1888, Tome Secund, Paris, 1890.

[11] O.Darrigol, *Between hydrodynamics and elasticity theory : the first five births of the Navier-Stokes equation*, Arch. Hist. Exact Sci., **56**(2002), 95-150.

[12] O.Darrigol, *Worlds of flow : a history of hydrodynamics from the Bernoullis to Prandtl*, Oxford Univ. Press, 2005.

[13] J.-B.-J. Fourier, *Théorie analytique de la chaleur. Deuxième Édition*, Paris, 1822. (This is available by G.Darboux [10] (Tome Premier) with comments).

[14] F.Graber, *Article ou Mémoire? Une réflexion comparative sur l'écriture des textes scientifiques, Navier et l'écoulement des fluides (1822-1827)*, Revue d'histoire des mathématiques, **10**(2004), 141-185.

[15] G.Green, *On the propagation of light in crystallized media*, Nottingham, 1839. (Edited by N.M.Ferrers, Ibid.)

[16] W.M.Hicks, *Report on recent progress in hydrodynamics*, British Association for the Advancement of Science, Report(1881), 57-88, (1882), 39-70.

[17] I.Imai, *Fluid dynamics*, Physical text series, Iwanami Pub. (1970), (New edition, 1997). (Japanese)

[18] J.L.Lagrange, *Méchanique analitique*, Paris, 1788. (Quatrième édition d'après la Troisième édition de 1833 publiée par M. Bertrand, *Joseph Louis de Lagrange, Oeuvres*, publièes par les soins de J.-A. Serret et Gaston Darboux, **11/12**, Georg Olms Verlag, Hildesheim·New York, 1973.) (J.Bertarnd remarks the differences between the editions.)

[19] J.L.Lagrange, *Méchanique analitique*, Paris, 1788. (Editions Jacques Gabay in 1989. première édition.)

[20] H.Lamb, *Hydrodynamics*, 1895, Cambridge.

[21] P.S.Laplace, *Traité de méchanique céleste*, Ruprat, Paris, 1798-1805.

[22] P.S.Laplace, *On capillary attraction, Supplement to the tenth book of the Méchanique céleste*, translated by N. Bowditch, same as above Vol. IV 685-1018.

[23] S.Masuda, *Constructions and developments of the solution on the Navier-Stokes equations around about Sobolev's embedding theorem. - 17th Symposium on the History of Mathematics*, Reports of Institute for Mathematics and Computer science **28**, Tsuda College, 2007, 327-350.

[24] S.Masuda, *Arguments among the Géomètres on the original Navier's equations in the prime of the second molecular period. - 18th Symposium on the History of Mathematics*, Reports of Institute for Mathematics and Computer science **29**, Tsuda College, 2008, 239-260.

[25] C.L.M.H.Navier, *Sur les lois du mouvement des fluides, en ayant égard à l'adhésion des molécules*, Ann. chimie phys., **19**(1822), 244-260.

[26] C.L.M.H.Navier, *Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques*, Mémoires de l'Academie des Sience de l'Institute de France, **7**(1827), 375-393. (Lu : 14/mai/1821.) → http://gallica.bnf.fr/ark:/12148/bpt6k32227, 375-393.

[27] C.L.M.H.Navier, *Mémoire sur les lois du mouvement des fluides*, Mémoires de l'Academie des Sience de l'Institute de France, **6**(1827), 389-440. (Lu : 18/mar/1822.) → http://gallica.bnf.fr/ark:/12148/bpt6k3221x, 389-440.

[28] C.L.M.H.Navier, *Note relative à l'article intitulè : Mémoire sur les équilibre et le mouvement des Corps élastiques, page 337 du tome précédent*, Annales de chimie et de physique, **38**(1828), 304-314.

[29] C.L.M.H.Navier, *Lettre de M.Navier à M.Arago*, Annales de chimie et de physique, **39**(1829), 99-107. (This is followed by) *Note du Rédacteur*, 107-110.

[30] C.L.M.H.Navier, *Note relative à la question de l'équilibre et du mouvement des corps solides élastiques*, Bulletin des sciences mathématiques, astromatiques, physiques et chimiques, **11**(1829), 249-253. (The title number : No.142.)

[31] I.Newton, *Philosophiae naturalis principia mathematica*, London, 1687. (Latin)

[32] J.J.O'Connor, E.F.Robertson,→ http://www-groups.dcs.st-and.ac.uk/ history/Printonly/Saint-Venant.html.

[33] S.D.Poisson, *Mémoire sur les intégrales définies*, (1813), J. école Polytech., Cahier**16**, **9**(1813), 215-246.

[34] S.D.Poisson, *Mémoire sur les Surfaces élastiques*, Institut National des Sciences et des Arts, Mémoires des siences mathematiques et physiques, **9**(1814), 167-226.

[35] S.D.Poisson, *Suite du Mémoire sur les intégrales définies, imprimé dans le volume précédent de ce Journal*, J. École Polytech., **10**(1815), 612-631.

[36] S.D.Poisson, *Suite du Mémoire sur les Intégrales définies, Inséré dans les duex précédens volumes de ce Journal*, J. école Polytech., **11** (1820), 295-341.

[37] S.D.Poisson, *Mémoire sur la Manière d'exprimer les Fonctions par des Séries de quantités périodiques, et sur l'Usage de cette Transformation dans la Résolution de différens Problèmes*, J. école Polytech., **11**(1820), 417-489.

[38] S.D.Poisson, *Extrait d'un Mémoire sur la Distribution de la chaleur dans les corps solides*, Annales de chimie et de physique, **19**(1821), 337-349. (Lu : 31/déc/1821.)

[39] S.D.Poisson, *Addition Au Mémoire precedent, et au Mémoire sur la manière d'exprimer les Fonctions par des Séries de Quantités périodiques*, J. école Polytech., **12**(1823), 145-162.

[40] S.D.Poisson, *Suite du Mémoire sur les Intégrales définies et sur la Sommation des Séries*, J. école Polytech., **12** (1823), cahier **19**, 405-509.

[41] S.D.Poisson, *Extrait d'un Mémoire sur la Propagation du mouvement dans les fluides élastiques*, Annales de chimie et de physique, 2^e Ser., **22**(1823), 250-269. (Lu : 24/mar/1823.)

[42] S.D.Poisson, *Mémoire sur le calcul numérique des intégrales définies*, Mémoires de l'Academie des Sience de l'Institute de France, **7**(1827), 571-604. (Lu : 11/déc/1826.) → http://gallica.bnf.fr/ark:/12148/bpt6k32227, 571-604.

[43] S.D.Poisson, *Mémoire sur l'équilibre et le Mouvement des Corps élastiques*, Annales de chimie et de physique, **37**(1828), 337-355.

[44] S.D.Poisson, *Lettre de M.Poisson à M.Arago*, Annales de chimie et de physique, **39**(1828), 204-211.

[45] S.D.Poisson, *Mémoire sur l'Équilibre et le Mouvement des Corps élastiques*, Mémoires de l'Academie royale des Siences, **8**(1829), 357-570, 623-27. (Lu : 14/apr/1828.) → http://gallica.bnf.fr/ark:/12148/bpt6k3223j

[46] S.D.Poisson, *Mémoire sur l'Équilibre des fluides*, Mémoires de l'Academie royale des Siences, **9**(1830), 1-88. (Lu : 24/nov/1828.) → http://gallica.bnf.fr/ark:/12148/bpt6k3224v, 1-88.

[47] S.D.Poisson, *Mémoire sur les équations générales de l'équiblibre et du mouvement des corps solides élastiques et des fluides*, (1829), J. École Polytech., **13**(1831), 1-174. (Lu : 12/oct/1829.)

[48] L.Prandtl, *Über Flüssigkeitsbewegung bei sehr kleiner Reibung*, in III. Internationaler Mathematiker-Kongress in Heiderrberg vom 8. bis 13. August 1904. *Verhundlungen*, A. Krazer (ed.) 184-91, Leibzig, 1905. Also : Ludwig Prandtl, *Gesammelte Abhundlungen zur Mecchanik, Hydro-und Aerodynamik*, 3 vols(1961), Göttingen. vol **2**, 575-584. (read 1904.)

[49] L.Prandtl, *Fundamentals of hydro-and aeromechanics*, McGrawhill, 1934. (Based on lectures of L.Prandtl (1929) by O.G.Tietjens, translated to English by L.Rosenhead. 1934.)

[50] Lord Rayleigh (William Strutt), *On the circulation of air observed in Kundt's tubes, and on the some allied acustical problems*, Royal Society of London, *Philosophical transactions*, also in Lord Rayleigh, *Scientific papers*, 1883, **2**, no.108, 239-257.

[51] A.J.C.B.de Saint-Venant, *Mémoire sur le calcul de la résistance et de la flexion des pièces solides à simple ou à double courbure, en prenent simultanément en considération les divers efforts auxquels elles peuvent étre soumises dans tous les sens. (Extrait par l'auteur.)*, Académie des Sciences, *Comptes-rendus hebdomadaires des séances*, **17**(1843), 942-954, 1020-1031.

[52] A.J.C.B.de Saint-Venant, *Note à joindre au Mémoire sur la dynamique des fluides. (Extrait.)*, Académie des Sciences, *Comptes-rendus hebdomadaires des séances*, **17**(1843), 1240-1243. (Lu : 14/apr/1834.)

[53] A.J.C.B. de Saint-Venant, *Note complémentaire sur le problème des mouvements que peuvent prendre les divers points d'un solide ductile ou d'un liquide contenue dans un vase, pendant son écoulement par un orifice inférieur*, Académie des Sciences, *Compte-rendus hebdomadaires des séances*, **67**(1868) 278-282.

[54] H.Schlichting, *Boundary-Layer Theory*, 6th ed., McGraw-Hill, 1968. (English version translated by J.Kestin) (The original title : *Grenzschicht-Theorie*, 1951 in German).

[55] G.G.Stokes, *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, 1849*, (read 1845), (From the *Transactions of the Cambridge Philosophical Society* Vol. VIII. p.287), Johnson Reprint Corporation, New York and London, 1966, *Mathematical and physical papers* **1**, 1966, 75-129, Cambridge.

[56] G.G.Stokes, *Report on recent researches on hydrodynamics*, *Mathematical and physical papers* **1**, 1966, 157-187, Cambridge.

[57] C.Truesdell, *Notes on the History of the general equations of hydrodynamics*, Amer. Math. Monthly **60**(1953), 445-458.

[58] C.Truesdell, *Leonhardi Euleri Opera Omnia. The rational fluid mechanics. 1687-1765. Introduction to Leonhard Euleri Opera Omnia. Vol XII seriei secundae*, Auctoritate et impensis societatis scientiarum naturalium helveticae, **12** 1954.

Remark : we use *Lu* (: in French) in the bibliography meaning "read" date by the judges of the journals, for example, MAS : *Mémoires des Académie des Sciences.* In citing the original paragraphs in our paper, the underscoring are by ours.

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