The core papers of Dieudonne theory on formal group laws

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As a vita, let us review Jean Dieudonne. He was born in Lille (Nord/Flandre) on July 1, 1906, and passed away on November 29, 1992. Though it is useless information, I would like to mention about the name " Dieudonne " .
A French word " Dieu " means " God ", and a French verb " donner " means " give ". From 1952 to 1959, he was visiting at Northwestern University in USA. In this period, he wrote eight series of papers on FGL (; formal group laws) over a field of positive characteristic, which are numbered from (I) to (VIII). The odd numbers of them (i.e. (I),(III),(V),(VII)) are written in French. The even numbers of them (i.e. (II),(IV),(VI),(VIII)) are written in English, and those were published in American Journal of Mathematics.

In the second volume of Dieudonne collected papers, we can find all of them except for (VI) and (VII). In order to recall the Dieudonne theory, we shall quote main theorems from his seventh paper (VII).

Théorème 1. Tout groupe abélien formel sur un corp. algébriquement clos est isogène à un produit direct de groupe de Witt et de groupe simple.

Théorème 2. Tout groupe abélien simple de dimension n sur un corps algébriquement clos est isogène à un groupe $G_{n,0,m}$, où m est un entier premier à n.

To prove the above theorems, he used several lemmas and propositions as follows.

Lemme 2. Pour qu'un homomorphisme v de G dans G' soit surjectif (resp. injectif) il faut et il suffit que l'homomorphisme correspondant de $F_0 = E_0/M_0$ dans $F_0' = E_0'/M_0'$ soit surjectif (resp. injectif).

Lemme 3. Si un $\mathcal{A}$-module monogène $\mathcal{A}/\mathcal{A}\mathfrak{a}$ ($\mathfrak{a} \neq 0$, $s(\mathfrak{a}) > 0$) est isomorphe à un $\mathcal{A}$-module E_0/M_0 associé à un groupe abélien formel, on a nécessairement pour une forme quasi-normale (1) (i.e. P.119 $\mathfrak{a} = \pi^{\delta_0} t^{\beta_0} a_0 + \pi^{\delta_1} t^{\beta_1} a_1 + \cdots + \pi^{\delta_r} t^{\beta_r} a_r$) de $\mathfrak{a}$, ou bien $r = 0$, ou bien $\beta_r - \beta_i \geqq \delta_i - \delta_r$ pour une valeur de i au moins ($0 \leqq i < r$).

Proposition 4. Soient G un groupe abélien formel, π l'endomorphisme « p-ème puissance » dans G ; G est produit quasi-direct du plus grand sous-groupe unipotent U de G et du plus grand sous-groupe T de G dans lequel π est une isogénie ; en outre U est isogène à un produit direct de groupes de Witt W_{m_i}.

Lemme 5. Si $\mathfrak{B}$ et $\mathfrak{C}$ sont deux éléments inversibles de $\mathcal{E}^+ = W(\mathfrak{k}, \sigma)$, les éléments $\pi^r - \mathfrak{t}^s\mathfrak{B}$ et $\pi^r - \mathfrak{t}^s\mathfrak{C}$ de $\mathfrak{A}$ sont semblables ; de façon précise, il existe $\mathfrak{X}$ inversible dans $\mathcal{E}^+$ tel que

$$(10) \qquad \mathfrak{X}(\pi^r - \mathfrak{t}^s\mathfrak{B}) = (\pi^r - \mathfrak{t}^s\mathfrak{C})\mathfrak{X}$$

et $\mathfrak{Y}$ inversible dans $\mathcal{E}^+$ tel que

$$(11) \qquad \mathfrak{t}^{-s}\mathfrak{Y}(\pi^r - \mathfrak{t}^s\mathfrak{B}) - (\pi^r - \mathfrak{t}^s\mathfrak{C})\mathfrak{t}^{-s}\mathfrak{Y} = 1.$$

Proposition 5. Soit $\mathfrak{a} = \pi^n a_0 + \pi^{\alpha_1} t^{\beta_1} a_1 + \cdots + \pi^{\alpha_{k-1}} t^{\beta_{k-1}} a_{k-1} + t^m a_k$ un élément de $\underline{E}^+$ non divisible par π, mis sous forme quasi-normale (a_i inversibles dans $\underline{E}^+$) ; pour que $\mathfrak{a}$ soit irréductible dans A, il faut et il suffit que n et m soient premiers entre eux et que l'on ait

$$\frac{\alpha_i}{n} + \frac{\beta_i}{m} > 1 \quad \text{pour } 1 \leq i \leq k-1 ;$$

$\mathfrak{a}$ est alors semblable à $\pi^n - t^m$.

Proposition 6. Tout A-module de torsion N de type fini et tel que $\pi N = N$ est semisimple.

Corollaire 3. Soient $(\mathfrak{a}_i)$, $(\mathfrak{b}_i)$ deux suite de q éléments de $\mathfrak{A}$ non multiples de π, tels que $\mathfrak{b}_i$ soit semblable à $\mathfrak{a}_i$ pour $1 \leqq i \leqq q$. Alors tout produit $\mathfrak{b}_{i_1}\mathfrak{b}_{i_2}\cdots\mathfrak{b}_{i_q}$ (où $(i_1,\cdots,i_q)$ est une permutation de $(1,2,\cdots,q)$ est semblable à $\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_q$.

Reference

Dieudonné, J. ; Groupes de Lie et hyperalgèbres de Lie sur un corps de charactéristique $p>0$ (VII), Math. Annalen, Bd. 134, S. 114-133 (1957).

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APPENDIX

Dieudonne-Betti numbers

In this appendix, we consider the number $\chi(F\times G)$ (; Euler-Poincaré number) of a direct product of formal group laws F and G . Let us recall the definition of Dieudonne typical subgroup laws. Let F (= F(X,Y)) be a n-dimensional naive formal group law over a field K with char(K) $\neq$ 2. In fact, F consists of n-tuple of formal power series $F_j(X,Y)$ with 2n variables $X = (X_1, X_2, \ldots, X_n)$,

$Y = (Y_1, Y_2, \ldots, Y_n)$ such that $F(F(X,Y),Z) = F(X, F(Y,Z))$ and $F(X,0) = F(0,X) = X$, where $0 = (0,0,\ldots,0)$. Then we write by $F(X,Y) = (F_j(X,Y))_{1\leq j\leq n}$.

Let S be a non-empty proper subset of $\{1,2,\ldots,n\}$, and write by $X'_j = X_j$ or 0 , in accordance with $j\in S$ or $j \notin S$, respectively.

Furthermore, we introduce the following notations ;

$$X' = (X'_j)_{1\leq j\leq n}, \quad X|S = (X'_j)_{j\in S} \quad \text{(; restriction)},$$

$$F|S = (F_j(X', Y'))_{j\in S} = (F|S)(X|S , Y|S) .$$

Then $F|S$ is called a Dieudonne typical subgroup of F (with respect to S) if and only if $F_i(X', Y') = 0$ for every $i \notin S$. This is the case, one sees that $\dim(F|S) = \#(S)$.

Let $t_j(F)$ (; the j-th Dieudonne-Betti number of F) be the number of j-dimensional Dieudonne typical subgroups of F. Here we know that $0 \leq t_j(F) \leq \binom{n}{j} = \frac{n!}{(j!)(n-j)!}$

, where $n = \dim(F)$, $1 \leq j \leq n-1$. Hence we can define Euler-Poincare number of F by using Dieudonne-Betti numbers as follows.

$$\chi(F) = \sum_{1 \leq j \leq n-1} (-1)^j \, t_j(F)$$

Proposition. Under the above notations, we have a following product formula ;

$$1 + \chi(F \times G) = \left\{ 1 + \chi(F) \right\} \left\{ 1 + \chi(G) \right\} + (-1)^{\dim(F)} \left\{ 1 + \chi(G) \right\} + (-1)^{\dim(G)} \left\{ 1 + \chi(F) \right\} .$$

Proof. For an arbitrary Dieudonne typical subgroup D of $F \times G$, the contribution of D to $\chi(F \times G)$ can be divided into five cases as follows.

(i) $0 \times D$, $D < G$; $\chi(G)$

(ii) $D \times 0$, $D < F$; $\chi(F)$

(iii) $D = D_1 \times D_2$, $D_1 < F$, $D_2 < G$; $\chi(F)\,\chi(G)$

(iV) $D = F \times D_2$, $D_2 < G$ or $D_2 = 0$; $(-1)^{\dim(F)} \left\{ 1 + \chi(G) \right\}$

(V) $D = D_1 \times G$, $D_1 < F$ or $D_1 = 0$; $(-1)^{\dim(G)} \left\{ 1 + \chi(F) \right\}$.

For example, lower-dimensional calculations of (iii) are carried out explicitly as follows. $(-1)^2\, t_2(F\times G) = (-1)^2\, t_2(F) + (-1)^2\, t_2(G) + (-1)^1 t_1(F)\, (-1)^1 t_1(G)$,

$$(-1)^3 t_3(F\times G) = (-1)^3 t_3(F) + (-1)^3 t_3(G) + (-1)^1 t_1(F)\, (-1)^2 t_2(G) + (-1)^2 t_2(F)(-1)^1 t_1(G) ,$$

$$(-1)^4 t_4(F\times G) = (-1)^4 t_4(F) + (-1)^4 t_4(G) + (-1)^1 t_1(F)\, (-1)^3 t_3(G) + (-1)^2 t_2(F)(-1)^2 t_2(G)$$
$$+ (-1)^3 t_3(F)\, (-1)^1 t_1(G) .$$

Thus we obtain our proposition. Q.E.D.

It follows from the above proposition that

$$\left\{ 1 + \chi(F) + (-1)^{\dim F} \right\} \left\{ 1 + \chi(G) + (-1)^{\dim G} \right\}$$

$$= 1 + \chi(F\times G) + (-1)^{\dim F + \dim G} .$$

Write by $\widetilde{\chi}(F) = \sum_{0\leq j\leq n} (-1)^j t_j(F)$, where $t_0(F) = t_n(F) = 1$, $n = \dim F$. Since $\dim(F\times G) = \dim F + \dim G$, we have a Whitney-type product formula as follows.

$$\widetilde{\chi}(F\times G) = \widetilde{\chi}(F)\, \widetilde{\chi}(G) .$$