Esquisse of a history of Dieudonné modules

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1. Ground Plan

Although the theory of Dieudonné modules was initiated by Jean Dieudonné in 1955, we have to make mention of Ernst Witt. In 1937, he introduced an important example of FGL (; formal group law) over a finite field $\mathbb{F}_q$, which is called the Witt group of Witt vectors with coefficients in $\mathbb{F}_q$. It influenced to the series of papers on FGL of Dieudonné. After the paper of Dieudonné (; Amer. J. Math. 77 (1955), 429–452), the following mathematicians have contributed in various aspects of the theory of Dieudonné modules ;

(1.1) I. Barsotti ; Abelian variety over $\mathbb{F}_q$

(1.2) M. Lazard ; Commutative formal group laws

(1.3) A. Grothendieck ; Elements of algebraic geometry, Dieudonné functor

(1.4) P. Cartier ; Isogeny of FGL and duality

(1.5) P. Gabriel ; Abelian category of Dieudonné

(1.6) Yu I. Manin ; Abelian FGL over $\mathbb{F}_q$

(1.7) J. Tate ; p-divisible groups

(1.8) T. Honda ; Logarithmic Dieudonné submodules

(1.9) P. Berthelot ; Crystalline cohomology

(1.10) J.-M. Fontaine ; Finite Honda Systems

Nowadays the theory of Dieudonné modules is called crystalline Dieudonné theory. Let D_k be the Cartier-Dieudonné ring over a field k. Since the Dieudonné modules are reduced D_k-modules, one briefly calls them Cartier modules. Nevertheless we should not forget that Cartier's three theorems were proved in fact by M. Lazard. In the above paper (P.448, #18), Dieudonné wrote as follows.

We thus see that, by associating with the group G the $\mathcal{E}^+(K)$-module E/M, the theory of abelian formal Lie groups over a perfect field K is essentially reduced to the theory of left $\mathcal{E}^+(K)$-modules.

2. Sections (Interior Elevations)

(2.1) Yoneda's lemma (; In homage to Nobuo Yoneda)

Around 1945, S. Eilenberg and S. MacLane introduced the concepts of category and functor in Algebraic topology. Then these concepts and terminologies were used in Homological algebra and Algebraic geometry because of its convenience.

Let $\mathcal{C} = (\mathcal{Ob}(\mathcal{C}), \mathcal{mor}(\mathcal{C}))$ be a category which consists of objects and morphisms. Write by

$$h^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y),$$

$$h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

for every objects X, Y in $\mathcal{Ob}(\mathcal{C})$.

Then h^X and h_X are functors from $\mathcal{C}$ to Ens (=Set ; the category of sets and maps).

Now, let F be a covariant functor from $\mathcal{C}$ to Ens. Then the correspondence $(x \longmapsto \mathcal{U}_x)$ defines a bijection between the set $F(X)$ and the set of functorial morphisms (i.e. natural transformations) from h^X to F. In fact, according to the lemma by Yoneda, we obtain that

$$
\begin{aligned}
F(X) &\xrightarrow{\;\approx\;} \mathcal{F}.\mathcal{M}.(h^X, F) \\
x &\longmapsto \mathcal{U}_x
\end{aligned}
$$

, where $(\mathcal{U}_x)_Y : h^X(Y) \longrightarrow F(Y)$ for every $Y \in Ob(\mathcal{C})$,

$$(\mathcal{U}_x)_Y(f) = F(f)(x) \text{ for every } f \in Hom(X,Y) = h^X(Y),$$

here $F(f) : F(X) \longrightarrow F(Y)$.

(2.2) Logarithm of a commutative FGL

Let $\mathbb{F}=\mathbb{F}(X,Y)$ be a 1-dimensional formal group law over $\mathcal{O}$. In fact, we have

$$\mathbb{F}(\mathbb{F}(X,Y),Z)=\mathbb{F}(X,\mathbb{F}(Y,Z)), \quad \mathbb{F}(X,0)=X,$$

$$\mathbb{F}(0,Y)=Y, \quad \text{and} \quad \mathbb{F}(Y,X)=\mathbb{F}(X,Y).$$

Then there exists a unique homomorphism $l_{\mathbb{F}}$ in $\mathrm{Hom}_{\mathcal{C}_{ab}(FGL)}(\mathbb{F},\mathbb{G}_a)$, where $\mathcal{C}_{ab}(FGL)$ denotes the category of commutative FGLs over $\mathcal{O}$, and $\mathbb{G}_a(X,Y)=X+Y$ denotes the additive FGL. This homomorphism $l_{\mathbb{F}}$ is called the logarithm of $\mathbb{F}$. Explicitly, we can write $l_{\mathbb{F}}$ as follows.

$$l_{\mathbb{F}}(X)=\int_0^X \frac{dX}{\left(\frac{\partial \mathbb{F}(X,Y)}{\partial X}\right)_{(0,X)}} = \int \omega_{\mathbb{F}}$$

, where $\omega_{\mathbb{F}}=\dfrac{dT}{\mathbb{F}_X(0,T)}$ denotes the invariant differential form of $\mathbb{F}$.

(2.3) Main theorems

Let $\mathcal{C}(FGL/k)$ be the category of formal group laws over a field k, and let $\mathcal{C}_{ab}(FGL/k)$ be its subcategory of commutative FGLs over k.

Let $\mathcal{C}(\text{Lie alg.}/k)$ be the category of Lie algebras over k, and $\mathcal{C}(\text{red.}{}_{D_k}M)$ be the category of reduced D_k-modules, where $D_k = \mathrm{Cart}(k)$ denotes the Cartier–Dieudonné ring of k. Then the following main theorems are well-known.

<u>THEOREM I</u>. If $\mathrm{char}(k)=0$, then we have a categorical equivalence as follows.

$$\mathcal{C}(FGL/k) \xrightarrow{\approx} \mathcal{C}(\text{Lie alg.}/k)$$

$$\mathbb{F} \longmapsto \mathrm{Lie}(\mathbb{F})$$

THEOREM II. If $\mathrm{char}(k) = p > 0$ and $\bar{k} = k$ (i.e. algebraically closed field), then we have a categorical equivalence as follows.

$$\mathcal{C}_{ab}(FGL/k) \xrightarrow{\approx} \mathcal{C}(\mathrm{red.}_{D_k}M)$$
$$\mathbb{F} \longmapsto M(\mathbb{F})$$

, where $M(\mathbb{F})$, whose definition will be described in (2.5), is called the Dieudonné module of $\mathbb{F}$.

(2.4) $\mathrm{Lie}(\mathbb{F})$

$x, y, z \in G$; a group

$x^y \underset{put}{=} y^{-1}xy$; a conjugate element of x

$(x^y)^z = x^{yz}$ $\quad (x \longmapsto x^y) \in \mathrm{Aut}(G)$

$(x, y) \underset{put}{=} x^{-1}y^{-1}xy$; commutator of x and y

Claim ; $(x^y,(y,z))(y^z,(z,x))(z^x,(x,y)) = 1 = e \in G$

identity ele.

<u>Proof.</u> $(x^y,(y,z)) = (x^y)^{-1}(y,z)^{-1}x^y(y,z)$

$= y^{-1}x^{-1}y\,(y^{-1}z^{-1}yz)^{-1}\,y^{-1}xy\,y^{-1}z^{-1}yz$

$= y^{-1}x^{-1}y\,z^{-1}y^{-1}zy\,y^{-1}x\,z^{-1}yz$

$= (yzy^{-1}xy)^{-1}zxz^{-1}yz$,

$(y^z,(z,x)) = (zxz^{-1}yz)^{-1}xyx^{-1}zx$,

$(z^x,(x,y)) = (xyx^{-1}zx)^{-1}yzy^{-1}xy$,

hence $\underset{\text{(left hand side)}}{LHS} = e$. Q.E.D.

A ; a Lie algebra over k

$\overset{\text{def.}}{\Longleftrightarrow}$ $\exists [\ ,\ ] : A \times A \longrightarrow A$ k-bilinear Lie bracket

such that (i) $[x,x] = 0$ for $\forall x \in A$

(ii) (Jacobi identity)

$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for $\forall x, \forall y, \forall z \in A$.

Let $\mathbb{F}=\mathbb{F}(\mathbb{X},\mathbb{Y})$ be a formal group law over a field k of characteristic 0. The order symbol $\mathcal{O}(\deg \geqq n)$ is used as follows.

$\mathcal{O}(\deg \geqq n) \equiv 0 \pmod{\deg n}$ (; it vanishes in homogeneous degree strictly less than n).

$$\underbrace{\mathbb{F}(\mathbb{X},\mathbb{Y})}_{\mathbb{X} \underset{\mathbb{F}}{\oplus} \mathbb{Y}} = \mathbb{X}+\mathbb{Y}+\underbrace{B(\mathbb{X},\mathbb{Y})}_{\text{Poly. term} (\deg=2)} + \mathcal{O}(\deg \geqq 3)$$

Write by $[\mathbb{X},\mathbb{Y}]_{\mathbb{F}} = B(\mathbb{X},\mathbb{Y}) - B(\mathbb{Y},\mathbb{X})$.

One sees that $\mathbb{O}=\mathbb{F}(\mathbb{X},\Psi(\mathbb{X}))$, where $\Psi(\mathbb{X}) = \sum_{j\geqq 1} \Psi_j(\mathbb{X})$ denotes the inverse of $\mathbb{X}$ with respect to $\mathbb{F}$, and $\Psi_j(\mathbb{X})$ is homogeneous of degree j ($j=1,2,3,\cdots$). Hence $\mathbb{O}=\mathbb{F}(\mathbb{X},\Psi(\mathbb{X})) = \mathbb{X}+\Psi_1(\mathbb{X})+\mathcal{O}(\deg \geqq 2)$ implies $\Psi_1(\mathbb{X}) = -\mathbb{X}$.

Furthermore $\mathbb{O} = \mathbb{F}(\mathbb{X}, \Psi(\mathbb{X})) = \mathbb{X} + \left(-\mathbb{X} + \Psi_2(\mathbb{X}) + \Psi_3(\mathbb{X}) + \cdots\right)$
$+ B(\mathbb{X}, -\mathbb{X} + \Psi_2(\mathbb{X}) + \Psi_3(\mathbb{X}) + \cdots) + \mathcal{O}(\deg \geqq 3)$
$= \Psi_2(\mathbb{X}) - B(\mathbb{X}, \mathbb{X}) + \mathcal{O}(\deg \geqq 3)$ implies $\Psi_2(\mathbb{X})$
$= B(\mathbb{X}, \mathbb{X})$.

Claim ; $[\mathbb{X}, \mathbb{Y}]_{\mathbb{F}}$ is a Lie bracket.
i.e. $[\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]_{\mathbb{F}}]_{\mathbb{F}} + [\mathbb{Y}, [\mathbb{Z}, \mathbb{X}]_{\mathbb{F}}]_{\mathbb{F}} + [\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]_{\mathbb{F}}]_{\mathbb{F}} = \mathbb{O}$.

<u>Proof</u>. Lemma (i) $\mathbb{X}^{\mathbb{Y}} = \mathbb{Y}^{-1} \mathbb{X} \mathbb{Y} = i(\mathbb{Y}) \underset{\mathbb{F}}{\oplus} \mathbb{X} \underset{\mathbb{F}}{\oplus} \mathbb{Y} = \mathbb{X} + \mathcal{O}(\deg \geqq 2)$,
(ii) $i(\mathbb{Y}) \underset{\mathbb{F}}{\oplus} i(\mathbb{Z}) \underset{\mathbb{F}}{\oplus} \mathbb{Y} \underset{\mathbb{F}}{\oplus} \mathbb{Z} = (\mathbb{Y}, \mathbb{Z}) = [\mathbb{Y}, \mathbb{Z}]_{\mathbb{F}} + \mathcal{O}(\deg \geqq 3)$,
(iii) $(\mathbb{X}^{\mathbb{Y}}, (\mathbb{Y}, \mathbb{Z})) = [\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]_{\mathbb{F}}]_{\mathbb{F}} + \mathcal{O}(\deg \geqq 4)$.

Hence
$$\mathbb{O} = (\mathbb{X}^{\mathbb{Y}}, (\mathbb{Y}, \mathbb{Z})) \underset{\mathbb{F}}{\oplus} (\mathbb{Y}^{\mathbb{Z}}, (\mathbb{Z}, \mathbb{X})) \underset{\mathbb{F}}{\oplus} (\mathbb{Z}^{\mathbb{X}}, (\mathbb{X}, \mathbb{Y}))$$
$$= [\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]_{\mathbb{F}}]_{\mathbb{F}} + [\mathbb{Y}, [\mathbb{Z}, \mathbb{X}]_{\mathbb{F}}]_{\mathbb{F}} + [\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]_{\mathbb{F}}]_{\mathbb{F}} + \mathcal{O}(\deg \geqq 4)$$
implies the above claim. Q.E.D.

(2.5) $D_k = \mathrm{Cart}(k)$ and $M(\mathbb{F})$

$\mathbb{F} = \mathbb{F}(\mathbb{X}, \mathbb{Y})$; n-dimensional commutative FGL over a ring A

$\mathbb{X} = (X_1, \cdots, X_n)$, $\mathbb{Y} = (Y_1, \cdots, Y_n)$; independent generic points of $\mathbb{F}$

$\mathbb{F}(\mathbb{F}(\mathbb{X}, \mathbb{Y}), \mathbb{Z}) = \mathbb{F}(\mathbb{X}, \mathbb{F}(\mathbb{Y}, \mathbb{Z}))$, $\mathbb{F}(\mathbb{0}, \mathbb{Y}) = \mathbb{Y}$,

$\mathbb{F}(\mathbb{X}, \mathbb{0}) = \mathbb{X}$, where $\mathbb{0} = (0, \cdots, 0)$, $\mathbb{F} = (F_1, F_2, \cdots, F_n)$,

and $\mathbb{F}(\mathbb{Y}, \mathbb{X}) = \mathbb{F}(\mathbb{X}, \mathbb{Y})$.

$\mathcal{C}(\mathbb{F})$; the module (i.e. abelian group) of curves in $\mathbb{F}$

$\mathcal{C}(\mathbb{F}) \ni \gamma = \gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$, where t denotes one-parameter.

$$\delta = \delta(t) = (\delta_1(t), \cdots, \delta_n(t)) \in \mathcal{C}(\mathbb{F})$$

$$\gamma + \delta \underset{def.}{=} \gamma \underset{\mathbb{F}}{\oplus} \delta = \mathbb{F}(\gamma(t), \delta(t))$$

The following operators of $\mathcal{C}(\mathbb{F})$ are important staffs of the Cartier-Dieudonné ring of $\mathbb{F}$.

$F_n,\ V_n,\ \langle a\rangle : \mathcal{C}(\mathbb{F}) \longrightarrow \mathcal{C}(\mathbb{F})$

, where $n \in \mathbb{N}$, $a \in A$.

$$F_n \gamma \underset{def.}{=} \gamma(\zeta_n t^{\frac{1}{n}}) + \gamma(\zeta_n^2 t^{\frac{1}{n}}) + \gamma(\zeta_n^3 t^{\frac{1}{n}}) + \cdots + \gamma(\zeta_n^n t^{\frac{1}{n}})$$

, where $\zeta_n = \exp(2\pi\sqrt{-1}/n)$ denotes the primitive n-th root of unity.

$$V_n \gamma \underset{def.}{=} \gamma(t^n) \qquad \langle a\rangle \gamma \underset{def.}{=} \gamma(at)$$

$$\mathbb{D}_A = \mathrm{Cart}(A) \underset{def.}{=} \left\{ \sum_{m,n \in \mathbb{N}} V_m \langle a_{m,n}\rangle F_n \ ; \ \begin{array}{l} a_{m,n} \in A \\ {}^{\forall} 1 \leqq m,\ a_{m,n} = 0 \text{ for} \\ \text{almost all } n \end{array} \right\}$$

is called the Cartier-Dieudonné ring over A.

Furthermore $\mathbb{D}_A$-module $\mathcal{C}(\mathbb{F})$ is called the Dieudonné module of $\mathbb{F}$, and we write it by $M(\mathbb{F})$.

The ring structure of $\mathbb{D}_A$ is written as follows.

$$\langle a\rangle\langle b\rangle = \langle ab\rangle \quad , \text{where } a, b \in A.$$

$$\langle 1_A\rangle = F_1 = V_1 = 1_{\mathbb{D}_A} \quad ; \text{identity operator in } \mathbb{D}_A$$

$$\langle a\rangle V_m = V_m \langle a^m\rangle$$

$$F_m \langle a\rangle = \langle a^m\rangle F_m$$

$$F_n V_n = n\, 1_{\mathbb{D}_A}$$

If $G.C.D(m,n)=1$, then $V_m F_n = F_n V_m$.

Verschiebung operators ; $V_m V_n = V_{mn}$

Frobenius operators ; $F_m F_n = F_{mn}$

(exercise!)

(2.6) Reduced module

First we have to define a uniform module. Let E be the Cartier-Dieudonné ring of a field K. A uniform E-module is a topological left E-module $\mathcal{C}$ having the following property;

for any indexed set $(x_j)_{j\in J}$ of elements converging towards 0 in $E = \mathrm{Cart}(K)$ $(x_j \to 0)$, and for any set $(\gamma_j)_{j\in J}$ in $\mathcal{C}$, the sum $\sum_{j\in J} x_j \gamma_j$ converges in $\mathcal{C}$.

A reduced E-module is a uniform E-module $\mathcal{C}$, satisfying the following three conditions;

(i) the topology of $\mathcal{C}$ is its $(\mathcal{C}_n)$-topology. Here $\mathcal{C}_n = \{\gamma \in \mathcal{C} \;;\; \gamma(t) \equiv 0 \ (\mathrm{mod.\ deg}\, n)\}$, and $\mathcal{C} = \mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_n \supset \mathcal{C}_{n+1} \supset \cdots$.

$$\mathcal{C} = \varprojlim_n \mathcal{C}/\mathcal{C}_n \quad , \quad gr_n(\mathcal{C}) \underset{def.}{=} \mathcal{C}_n/\mathcal{C}_{n+1}$$

(ii) the map $gr_1(\mathcal{C}) \longrightarrow gr_m(\mathcal{C})$ induced by V_m is bijective, for any $m \in \mathbb{N}$.

(iii) the K-module $gr_1(\mathcal{C})$ is free.

Then it is well-known that the module of curves $\mathcal{C}(\mathbb{F}) \left(= M(\mathbb{F})\right)$ is a reduced module.

(2.7) Witt ring

Let p be a rational prime, and let A be a K-algebra over a field K. For independent infinite variables $\mathbb{X}=(X_0, X_1, X_2, \cdots)$, $\mathbb{Y}=(Y_0, Y_1, Y_2, \cdots)$, the Witt polynomials $W_n(\mathbb{X})$ $(n=0,1,2,\cdots)$ are defined as follows.

$$W_0 = X_0 \qquad , \quad W_1 = X_0^p + pX_1 ,$$

$$W_2 = X_0^{p^2} + pX_1^p + p^2 X_2 ,$$

$$W_3 = X_0^{p^3} + pX_1^{p^2} + p^2 X_2^p + p^3 X_3 ,$$

$$\vdots$$

$$W_n = \sum_{0 \leqq j \leqq n} p^j X_j^{p^{n-j}} .$$

The addition is defined inductively as follows.

$$W_n(Z_0, Z_1, Z_2, \cdots) = W_n(X_0, X_1, X_2, \cdots) + W_n(Y_0, Y_1, Y_2, \cdots)$$

$$\Longleftrightarrow \quad Z_0 = X_0 + Y_0 \ , \quad Z_0^p + pZ_1 = X_0^p + pX_1 + Y_0^p + pY_1 \ ,$$

$$Z_0^{p^2} + pZ_1^p + p^2 Z_2 = X_0^{p^2} + pX_1^p + p^2 X_2 + Y_0^{p^2} + pY_1^p + p^2 Y_2 \ , \cdots$$

Hence $\quad Z_1 = X_1 + Y_1 + p^{-1}\{X_0^p + Y_0^p - (X_0 + Y_0)^p\}$, $Z_2 = X_2 + Y_2 + p^{-1}X_1^p + p^{-1}Y_1^p + p^{-2}X_0^{p^2} + p^{-2}Y_0^{p^2} - p^{-2}(X_0 + Y_0)^{p^2} - p^{-1}\{X_1 + Y_1 + p^{-1}(X_0^p + Y_0^p) - p^{-1}(X_0 + Y_0)^p\}^p$, $\cdots$.

The multiplication is defined inductively as follows.

$$W_n(Z_0, Z_1, Z_2, \cdots) = W_n(X_0, X_1, X_2, \cdots)\, W_n(Y_0, Y_1, Y_2, \cdots)$$

$$\Longleftrightarrow \quad Z_0 = X_0 Y_0 \ , \quad Z_0^p + pZ_1 = (X_0^p + pX_1)(Y_0^p + pY_1) \ ,$$

$$Z_0^{p^2} + pZ_1^p + p^2 Z_2 = (X_0^{p^2} + pX_1^p + p^2 X_2)(Y_0^{p^2} + pY_1^p + p^2 Y_2) \ , \cdots$$

Hence $\quad Z_1 = X_1 Y_0^p + X_0^p Y_1 + pX_1 Y_1$, $Z_2 = p^2 X_2 Y_2 + X_1^p Y_1^p + p^{-1}X_0^{p^2} Y_1^p + X_0^{p^2} Y_2 + p^{-1}X_1^p Y_0^{p^2} + pX_1^p Y_2 + X_2 Y_0^{p^2} + pX_2 Y_1^p - p^{-1}(X_1 Y_0^p + X_0^p Y_1 + pX_1 Y_1)^p$, $\cdots$.

Thus we can define an infinite dimensional Witt ring $W(A)$ over A. In fact, we have used the following theorem ;

<u>THEOREM</u>. For every polynomial Φ in $\mathbb{Z}[X,Y]$, there uniquely exists a series $(\varphi_0, \varphi_1, \varphi_2, \cdots)$ of elements φ_n in $\mathbb{Z}[X_0, X_1, X_2, \cdots; Y_0, Y_1, Y_2, \cdots]$ $(n=0,1,2,\cdots)$ such that

$$W_n(\varphi_0, \varphi_1, \varphi_2, \cdots) = \Phi\left(W_n(X_0, X_1, X_2, \cdots), W_n(Y_0, Y_1, Y_2, \cdots)\right)$$

for every $n \geqq 0$.

3 Epilogue

The following pair of words by Kierkegaard in "The sickness unto death" reminds us the birth of mathematics.

α. Infinitude's despair is to lack finitude.

β. Finitude's despair is to lack infinitude.

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