CONSTRUCTIONS AND DEVELOPMENTS OF THE SOLUTION ON THE NAVIER-STOKES EQUATIONS AROUND ABOUT SOBOLEV'S EMBEDDING THEOREM

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ABSTRACT. We introduce the formulation and the successions of the constructions of the solutions on the Navier-Stokes equations earlier on in the history until 1950s, summarizing with the following 4 types of these successions :

1) for the classical solutions, to formulate or rediscover or re-derive the Navier-Stokes equations by Newton,Bernoulli,D'Alembert,Euler, Lagrange,Laplace,Navier,Cauchy,Poisson,Saint-Venant,Stokes,
2) for the fundamental solutions, owing to Newton's potential theory, to construct the invariant tensor t_{ij} by Poisson,Cauchy,Green,Stokes,Oseen,Lichtenstein,Odqvist,Leray,Ladyzhenskaya,
3) for the Cauchy problem/turbulent solution/weak solutions to define and construct the conception/notion of the solution by Cauchy,Kovalevskaya,Hadamar,Leray,Hopf, and
4) for the generalized solutions/strong solutions, in using the functional analysis, especially, directly the Sobolev's tools, to construct the proof and reguraliry by Sobolev,Kiselev,Ladyzhenskaya,Prodi,J.L.Lions,Serrin.

1. EARLIER-ON SUCCESSIONS OF THE CONSTRUCTIONS OF THE SOLUTIONS ON NAVIER-STOKES EQUATIONS

For convenience' sake, we summarize the successions of the constructions of the solutions on Navier-Stokes equations, earlier on in the history until 1950s, with the following 4 types of the successions :

1) for the classical solutions, to construct the formulation or rediscovery or re-derivation[1] of Navier-Stokes equations,
2) for the fundamental solutions, owing to Newton's potential theory, to construct the invariant tensor t_{ij},
3) for the Cauchy problem/turbulent solution/weak solutions/ to construct the conception/notion of the solution.
4) for the generalized solutions[2]/strong solutions[3], in using the functional analysis, especially, directly the Sobolev's tools, to construct the proof and regurality.

We show four earlier-on successions of the constructions of the solutions of the Navier-Stokes equations in Table 1 and 2, and the successions of the invariant tensor : t_{ij} in Table 4,5 and 6.[4]

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[1]We owe to O.Darrigol [15] in view of the rediscovery or the re-derivation of Navier-Stokes equations.

[2]cf. We cite below the definition of the generalized solution with Kiselev & Ladyzhenskaya[34]'s formulations in §5.

[3]There is the theorem on the strong solution by the modern definition in the following :
Theorem (Kato, Giga) : $a \in L^n_\sigma \cap L^q_\sigma$, $n \le q < \infty \Rightarrow^\exists T > 0$, $^{\exists 1} u$: a strong solution of the Navier-Stokes equations on $(0,T)$ in the class : $u \in C((0,T); L^n_\sigma \cap L^q_\sigma) \cap C((0,T); W^{2,q}) \cap C^1((0,T); L^q_\sigma))$. The word : "strong" means $p = 2 > 1$ in $W^{p,q}$.

[4]In Odqvist[57], he uses "Der in der Flüssigkeit wirkend Spannungstensor", which means the stress tensor operating in the fluid.

TABLE 1. No.1-1 Four earlier-on successions of the constructions of the solution of the Navier-Stokes equations

no	1.formulation of equations	2.construction of tensor and integral equ.
	classical solution	fundamental solution
	formulation & rediscovery	invariant tensor : t_{ij}
	potential theory	potential theory
1	I.Newton[55](1643),[56](1687)	A.Cauchy[6](1828)
2	D.Bernoulli[2](1727),[3]('38)	S.D.Poisson[60](1829-31)
3	J.L.d'Alembert[13](1752)	G.G.Stokes[65](1845)
4	L.Euler[18](1752-55),[19](55),[20](60-61)	C.W.Oseen[59](1927)
5	J.L.Lagrange[39, 40](1788)	L.Lichtenstein[45](1928),[46]('29),[47]('31)
6	M.Laplace[41](1798-1805)	F.K.G.Odqvist[57](1930)[58]('32)
7	C.L.M.H.Navier[51](1822),[52, 53]('27)	J.Leray[42](1933),[43, 44]('34)
8	A.Cauchy(1823)	O.A.Ladyzhenskayay[36](1959),[37](1970)
9	S.D.Poisson[60](1829-31)	V.A.Solonnikov[64](1977)
10	Saint-Venant(1837)	
11	G.G.Stokes[65](1845)	

TABLE 2. No.1-2 Four earlier-on successions of the constructions of the solution of the Navier-Stokes equations

no	3.notion/conception of solution	4.functional analysis
	turbulent solution/weak solution	generalized solution/strong solution
		Sobolev's embedding theorem, etc
	potential theory	
1	A.Cauchy[7, 8, 9, 10, 11, 12](1842)	S.L.Sobolev[63](1950)
2	S.Kovalevskaya[35](1875)	A.A.Kiselev[32](1955),[33]('56)
3	G.Darboux[14](1875)	A.A.Kiselev& O.A.Ladyzhenskaya[34]('57)
4	J.Hadamard[24](1932)	O.A.Ladyzhenskaya[36](1959)
5	J.Leray[42](1933),[43, 44]('34)	G.Prodi[61](1959)
6	E.Hopf[28](1951)	J.L.Lions&G.Prodi[49](1959)
7		J.L.Lions[48](1959)
8		J.Serrin[62](1959)
9		
10		
11		

2. THE SUCCESSIONS OF THE FORMULATION OF THE NAVIER-STOKES EQUATIONS

2.1. **Euler's formulation.** In *"Sectio secunda de principiis motus fluidorum"* (The chapter 2 *on the priciple of the motion of the fluid*) [18], Euler shows the Euler's equations what we called today, in modern vectorial expression :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{1}$$

We show Euler's text [18, pp.85-95] as follow :[5]

> [**Problem 20**] **§19** From the givin three velocities u, v and w from which the individual elements of a fluid are moving, investigate the accelaration in an arbitrary interval of the infinitely short time : dt.
>
> [**Solution**] Considering (Fig.24), the elements of the fluid now transient at the point Z, determinated by the coordinates : $OX = x$, $XY = y$, $YZ =$

[5]This English version of L.Euler[18] is translated from Latin by Shigeru Masuda.

z, from which with the velocities u, v and w, the body arrive by the time dt to the interval z. Hence, this point is, by the three coordinates

$$Ox = x + udt, \quad xy = y + vdt \text{ and } yz = z + wdt$$

determinated. This position we mention is seeked by the three velocities, of which now the element at z exists and that are u', v', w', which surpass its three velocities u, v, w, how these turn out at Z? in this time, from this increments, the accelarations are estimated. Now, u, v and w are the functions of the four variables x, y, z and t, the velocities we mention in z, the body of the timespan dt, therefore are constructed, if the variables x, y, z and t of these increments udt, vdt, wdt and dt are added : and therefore it turns out

$$\begin{cases} u' = u + udt\left(\frac{du}{dx}\right) + vdt\left(\frac{du}{dy}\right) + wdt\left(\frac{du}{dz}\right) + dt\left(\frac{du}{dt}\right) \\ v' = v + udt\left(\frac{dv}{dx}\right) + vdt\left(\frac{dv}{dy}\right) + wdt\left(\frac{dv}{dz}\right) + dt\left(\frac{dv}{dt}\right) \\ w' = w + udt\left(\frac{dw}{dx}\right) + vdt\left(\frac{dw}{dy}\right) + wdt\left(\frac{dw}{dz}\right) + dt\left(\frac{dw}{dt}\right). \end{cases}$$

which in the motion of the investigation of the incremental velocities with the divided timespan get the accelaration, the three accelarations we mention, in this manner, we will estimate it :

$$\begin{cases} \frac{u'-u}{dt} = u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right) + \left(\frac{du}{dt}\right) \\ \frac{v'-v}{dt} = u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right) + \left(\frac{dv}{dt}\right) \\ \frac{w'-w}{dt} = u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right) + \left(\frac{dw}{dt}\right). \end{cases}$$

[**Problem 21**] §**23** If the initial three velocities u, v and w, which the individual points in the space by which the fluid is moved, it is suitable that each density q in an arbitrary point given, then investigate the relation which intervene between the velocities and densities.

[**Solution**] From the Problem 19, it turns out, if the fluid particles of these velocities u, v, w from Z to z the time dt, we put each density at Z $= q$, and that at $z = q'$, then it turns out

$$\frac{q'-q}{qdt} = -\left(\frac{du}{dx}\right) - \left(\frac{dv}{dy}\right) - \left(\frac{dw}{dz}\right).$$

Now, on the other hand, which density q such function given by the four variables x, y, z and t, expressed and now which particle at the point Z verified density denoted from it collected density q', if the body, time at the interval Z transfered so the four variable x, y, z and t these increments udt, vdt, wdt and dt are yielded true. Hence these densities q' of each particle from Z to z the translation is suitable thus to be expressed :

$$q' = q + udt\left(\frac{dq}{dx}\right) + vdt\left(\frac{dq}{dy}\right) + wdt\left(\frac{dq}{dz}\right) + dt\left(\frac{dq}{dt}\right).$$

and it turns out

$$\frac{q'-q}{dt} = u\left(\frac{dq}{dx}\right) + v\left(\frac{dq}{dy}\right) + w\left(\frac{dq}{dz}\right) + \left(\frac{dq}{dt}\right),$$

TABLE 3. Point and pressure

point	pressure	point	pressure
Z	p	z	$p+dz(\frac{dp}{dz})$
L	$p+dx(\frac{dp}{dx})$	l	$p+dx(\frac{dp}{dx})+dz(\frac{dp}{dz})$
M	$p+dy(\frac{dp}{dy})$	m	$p+dy(\frac{dp}{dy})+dz(\frac{dp}{dz})$
N	$p+dx(\frac{dp}{dx})+dy(\frac{dp}{dy})$	n	$p+dx(\frac{dp}{dx})+dy(\frac{dp}{dy})+dz(\frac{dp}{dz})$

which is able to valuate, if the above equation is substituted by the gained equation between the velocity and the density, from this equation it turns out as follows :

$$q\left(\frac{du}{dx}\right)+q\left(\frac{dv}{dy}\right)+q\left(\frac{dw}{dz}\right)+u\left(\frac{dq}{dx}\right)+v\left(\frac{dv}{dy}\right)+w\left(\frac{dw}{dz}\right)+\left(\frac{du}{dt}\right)=0,$$

here, because $q\left(\frac{du}{dx}\right)+u\left(\frac{dq}{dx}\right)=\left(\frac{d\cdot qu}{dx}\right)$, it turns out in short :

$$\left(\frac{dq}{dt}\right)+\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{d\cdot qv}{dy}\right)+\left(\frac{d\cdot qw}{dz}\right)=0,$$

which is obvious that the derivative of the qu is only with respect to x, that of the qv is only with respect to y and that of the qw is only with respect to z and then, the variable must be obeyed.

[**Problem 22**] §28 If the fluid moving by an arbitrary force from the-pressure in the individual points with gained known observed, investigate the force of the accelaration.

[**Solution**] According to (Fig.25), on the point by the orthogonal coordinate, $OX=x$, $XY=y$, $YZ=z$, the element of the fluid, now arround Z in the given figure of the parallel hexahedron, the rectangular $ZLMNzlmn$, we put the differential coordinate $ZL=dx$, $ZM=dy$, $Zz=dz$ continuous, with respect to which, the volume is put $=dxdydz$, as q is the density at Z, of which the mass is $=qdxdydz$. Now at first, we consider the similar forces of the gravitation in each direction at the watching point Z, by the direction axes OA, OB, OC, explain clealy, this accelarated force along the OA or when $ZL=P$, along the OB or when $ZM=Q$, along the OZ or when $Zz=R$, with which then the element of the accelarated fluid along each direction indicated, for which, in fact, it is not necessary for certain figure to attribute the element. In fact these figures, in paticular, are suitable to the forces of the accelaration, caused by the pressure. Therefore for this time, the height of the pressure in Z, given by $=p$, such that the function with the four variables x, y, z and t, situated as known : from which the natural pressure in the individual angles of the parallel hexahedron as defined, it turns out : (see Table.4) which pressure move in the individual normal plane.

Secondly, we consider the opposite surfaces $ZMzm$ and $LNln$ as well as the pressure is given, of which surface $LNln$ sustains the above pressure in each pressure element $dx(\frac{dp}{dx})$ by the opposite surface $ZMzm$, which single degeneration are gained in computation. Sustain this surface $LNln$ of the height of the pressure $dx(\frac{dp}{dx})$ is given ; and this area of the surface $=dydz$, the total pressure equals to the weight of the volume $dxdydz(\frac{dp}{dx})$, if it is certain that the material is homogeneous of which density $=1$, filled, which we assumed : and this direction of the force will parallels the axis AO, because it is normal on the surface. Because our hexahedron, of which mass $= qdxdydz$, it is moved to the next direction along AO the force of the motion $= dxdydz(\frac{dp}{dx})$, these are exactly derived by the mass derived will give the force of the accelaration $= \frac{1}{q}(\frac{dp}{dx})$; similary we calculate to gain the force of the accelaration, from which of our hexahedron of the next direction BO becomes $= \frac{1}{q}(\frac{dp}{dy})$, and the next direction $CO = \frac{1}{q}(\frac{dp}{dz})$. After all, these forces are contrary to them, which individual fluid are assumed, the elements of the fluid at Z sustain the following three forces of the accelaration

$$\begin{cases} \text{next direction } OA = P - \frac{1}{q}(\frac{dp}{dx}), \\ \text{next direction } OB = Q - \frac{1}{q}(\frac{dp}{dy}), \\ \text{next direction } OC = R - \frac{1}{q}(\frac{dp}{dz}). \end{cases}$$

We see that in which in any case of the forces, which the elements of the fluid. Even if, in fact, at any point, the fluid without operate to be pushed, hence the another force in the element is not preserved, unless by pressure p, which this is, as we have computated in above.

[**Problem 23**] **§33** If the fluid of the arbitrary nature is moved by the arbitrary force, under the stable initial condition, then from which, determine this motion clearly.

[**Solution**] Considering (fig.22), the stable fluid, in which the arbitrary temprature $= t$, suppose it is smooth, and the 3 dimensions of the fixed axes : OA, OB, OC, to those the normal directions, we consider the arbitrary particle of the fluid at the point of Z, which is fixed by the 3 coordinates : $OX = x$, $XY = y$, $YZ = z$, determinated and which is consisted of the accelarated forces P, Q, R, the following direction Zx, Zy, Zz by the axes of the given and of the parallel. First, to the motion of the fluid we mention, the stable, initial density of the particle, here in Z, supposed as $= q$, which we observe is expressed by such as the function of the four variables x, y, z and t. Second, now we put the pressure in Z the given value $= p$, which is permanent with respect to the material, uniformal gravity, of which density $= 1$ and is given; we write this p with the function of the four variables x, y, z and t. Third, every motion of the particles in Z, which we employ and observe here is determined by three directions Zx, Zy, Zz, of which velocities in accordance with $Zx = u$, $Zy = v$ and $Zz = w$, we explain considering of which velocities passing the space a little such as time t in the sequential time. Now, this situation is observed by the velocities and the density q, and this relation is determinated such that :[6]

$$\left(\frac{dq}{dt}\right) + \left(\frac{d \cdot qu}{dx}\right) + \left(\frac{d \cdot qv}{dy}\right) + \left(\frac{d \cdot qw}{dz}\right) = 0:$$

Therefore, in the above problem, we deduce the element of the fluid in Z, now this accelarated force as follows :

$$next.Zx = P - \frac{1}{q}\frac{dp}{dx}, \quad next.Zy = Q - \frac{1}{q}\frac{dp}{dy}, \quad next.Zz = R - \frac{1}{q}\frac{dp}{dz}. \tag{2}$$

[6]If, q is constant, this equation means div $\mathbf{u} = 0$.

From itself another motion of this three elements in problem 20, each accelaration of the each sequential direction, deduces in this manner, the expression :

$$\begin{cases} next\ Zx = u(\frac{du}{dx}) + v(\frac{du}{dy}) + w(\frac{du}{dz}) + (\frac{du}{dt}), \\ next\ Zy = u(\frac{dv}{dx}) + v(\frac{dv}{dy}) + w(\frac{dv}{dz}) + (\frac{dv}{dt}), \\ next\ Zz = u(\frac{dw}{dx}) + v(\frac{dw}{dy}) + w(\frac{dw}{dz}) + (\frac{dw}{dt}). \end{cases} \tag{3}$$

To which, here, substitute by this body's gravity a little moment, slide, weight $= g$ and time and the velocity of the sequencial of the above mentioned measure explain, an arbitrary accelarated from the force in $2g$ accelarated force, deduced smoothly, and we get the three following equations :

$$\begin{cases} 2gP - \frac{2g}{q}\left(\frac{dp}{dx}\right) = u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right) + \left(\frac{du}{dt}\right), \\ 2gQ - \frac{2g}{q}\left(\frac{dp}{dy}\right) = u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right) + \left(\frac{dv}{dt}\right), \\ 2gR - \frac{2g}{q}\left(\frac{dp}{dz}\right) = u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right) + \left(\frac{dw}{dt}\right). \end{cases}$$

which from the consideration of the density, the motion made by the continuous, universal, determinated continuous.

Here, we get (1) from (2) and (3).

2.2. **Navier's formulation.**

2.2.1. *Navier's principles.*

Navier ([52, pp.389-390]) says :

> By the partial differential equations, the geometry represents the general conditions of the equilibrium and of the motion of the fluid. These equations are deduced from the various principles which suppose the all which the molecule of the fluid are susceptible to take the one with reponce to the other of an arbitrary motion, without oppose any resistance and to slide without effort on the boundary of the vessel of which the fluid is contained. But the considerable or total differences, present, in the certain case, the natural effects with the result of the known theories indicate the necessary for the adoption of **the new notion** and to consider the certain action of the molecule which appears principally in the phenomena of the motion. We know, for example, that, in the case where the water in the vessel through the long pipe of the small diameter, the computation contribute to the fluid of the velocity of the flow which surpass it much, which we observe, and which is controlled from the differential law.

Navier cites the Euler's equations ([52, p.399]) :

$$\begin{cases} P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right), \\ Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right), \\ R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right), \end{cases}$$

$$0=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}. \tag{4}$$

Here, Navier says his paper's purpose as follows :

> But, owing to the notation state as follows, it is necessary to admit the existence of the *new molecular force*, which is developed by the state of the motion of the fluid. The study of the analytical equations of the force is **the main object** which we have intended in the composition of this paper. ([52, p.399])

In [51, p.252], owing to D'Alembert and Euler, Navier used his own equations published in 1821 ([51, p.250])[7]but which seem not to be the formation such today's equations as what we called Navier-Stokes equations on the incompressible fluid(7). His equations are as follw :

$$\begin{cases}\frac{1}{\rho}\frac{dp}{dx}=X+\varepsilon\Big(3\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+2\frac{d^2v}{dxdy}+2\frac{d^2w}{dxdz}\Big)-\frac{du}{dt}-\frac{du}{dx}\cdot u-\frac{du}{dy}\cdot v-\frac{du}{dz}\cdot w\ ;\\ \frac{1}{\rho}\frac{dp}{dy}=Y+\varepsilon\Big(\frac{d^2v}{dx^2}+3\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+2\frac{d^2u}{dxdy}+2\frac{d^2w}{dydz}\Big)-\frac{dv}{dt}-\frac{dv}{dx}\cdot u-\frac{dv}{dy}\cdot v-\frac{dv}{dz}\cdot w\ ;\\ \frac{1}{\rho}\frac{dp}{dz}=Z+\varepsilon\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+3\frac{d^2w}{dz^2}+2\frac{d^2u}{dxdz}+2\frac{d^2v}{dydz}\Big)-\frac{dw}{dt}-\frac{dw}{dx}\cdot u-\frac{dw}{dy}\cdot v-\frac{dw}{dz}\cdot w\ ;\end{cases} \tag{5}$$

and the equation of continuity (4) where ε is sensibly independ of the force which compress the partial diffrential of the fluid. Maybe in 1821, he was in his experimantal stage to formulate the Navier's equations. We can not substitute the operator Δ, which means $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, for (5), because the second terms in the right hand side are more than Δu with :

$$2\Big(\frac{d^2u}{dx^2}+\frac{d^2v}{dxdy}+\frac{d^2w}{dxdz}\Big),\quad 2\Big(\frac{d^2v}{dy^2}+\frac{d^2u}{dxdy}+\frac{d^2w}{dydz}\Big),\quad 2\Big(\frac{d^2w}{dz^2}+\frac{d^2u}{dxdz}+\frac{d^2v}{dydz}\Big). \tag{6}$$

In modern notation, the kinetic equation and the equation of continuity are conventionally described as follows :

$$\partial\mathbf{u}/\partial t-\mu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f},\quad \text{div }\mathbf{u}=0. \tag{7}$$

Navier says citing Laplace (*Equilibrium of Fluid* [41, Vol.1, Chap. 4-8, p.90-239]) :

> The consideration of the repulsive force, which the pressure develoves between the molecules, which M. de Laplace deduced already the general equations of the motion of the fluid in the 12-th book of *Méchanique céleste*, seems to depend more immidiately on the physical notion which we can formulate on the property of this corps.

Navier ([52, p.414]) had described the equation samely as today's vectotial expression (7) above stated as follows :
from (4), after operating in such a way as, at first by $\frac{d}{dx}$, and by $\frac{d}{dy}$, and at last by $\frac{d}{dz}$,

[7]Navier cited his paper as follows : dans un Mémoire sur les lois de l'équilibre et des mouvemens des corps solides élastiques, que j'ai présenté, le 14 mai 1821 (*sic.*). This title is none in Graber's citation [21].

then[8]

$$\begin{cases} \frac{d^2u}{dx^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz} = 0, \\ \frac{d^2v}{dy^2} + \frac{d^2u}{dxdy} + \frac{d^2w}{dydz} = 0, \\ \frac{d^2w}{dz^2} + \frac{d^2u}{dxdz} + \frac{d^2v}{dydz} = 0, \end{cases} \tag{8}$$

therefore (5) turns out :

$$\begin{cases} P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) - \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \\ Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) - \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right), \\ R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) - \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right). \end{cases} \tag{9}$$

and the equation of continuity which is the same as (4). Here, if we take $\mathbf{f} = (P, Q, R)$ and $\frac{1}{\rho}\mathbf{f} \equiv \mathbf{f}$, then this means :

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\varepsilon}{\rho}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}. \tag{10}$$

If we put $\mu \equiv \frac{\varepsilon}{\rho}$ then (9) equals to (7).

2.2.2. *Navier's deduction of the expression of forces of the molecular action which is under the state of motion.*

Navier deduce the expression of forces of the molecular action which is under the state of motion as follow in ([52, pp.399-405]): We consider the two molecules M and M'. x, y, z are the values of the rectanglar coordinates of M and $x+\alpha, y+\beta, z+\gamma$ are the values of the rectanglar coordinates of M'. $\rho = \sqrt{\alpha^2+\beta^2+\gamma^2}$. The velocity of the molecule M are u, v, w and that of the molecules M' are

$$u + \frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma, \quad v + \frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma, \quad w + \frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma \tag{11}$$

V is the quality on which the propotional action depends as follows :

$$V = \frac{\alpha}{\rho}\left(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\right) + \frac{\beta}{\rho}\left(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\right) + \frac{\gamma}{\rho}\left(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\right).$$

The increment of V is as follows :

$$\delta V = \frac{\alpha}{\rho}\left(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\right) + \frac{\beta}{\rho}\left(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\right) + \frac{\gamma}{\rho}\left(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\right).$$

$f(\rho)$ is a function depends on the distance ρ between M and M'.

$$f(\rho)V\delta V =$$

$$\frac{f(\rho)}{\rho^2}\left\{\ \alpha\left(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\right) + \beta\left(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\right) + \gamma\left(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\right)\ \right\}$$
$$\left\{\ \alpha\left(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\right) + \beta\left(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\right) + \gamma\left(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\right)\ \right\},$$

[8] [52, p.413]

here, by symmetry

$$\alpha\frac{du}{dy}\beta+\beta\frac{dv}{dx}\alpha=0,\quad \beta\frac{dv}{dz}\gamma+\gamma\frac{dw}{dy}\beta=0,\quad \alpha\frac{du}{dz}\gamma+\gamma\frac{dw}{dx}\alpha=0,$$
$$\alpha\frac{\delta du}{dy}\beta+\beta\frac{\delta dv}{dx}\alpha=0,\quad \beta\frac{\delta dv}{dz}\gamma+\gamma\frac{\delta dw}{dy}\beta=0,\quad \alpha\frac{\delta du}{dz}\gamma+\gamma\frac{\delta dw}{dx}\alpha=0$$

Because we integrate $\frac{1}{8}$ of the sphere, total is to be multiplied by 8.

$$8\frac{f(\rho)}{\rho^2}\begin{cases}\left(\frac{du}{dx}\frac{\delta du}{dx}\alpha^4+\frac{du}{dy}\frac{\delta du}{dy}\alpha^2\beta^2+\frac{du}{dz}\frac{\delta du}{dz}\alpha^2\gamma^2\right)+\\ \left(\frac{dv}{dy}\frac{\delta du}{dx}+\frac{dv}{dx}\frac{\delta du}{dy}\right)\alpha^2\beta^2+\\ \left(\frac{dw}{dz}\frac{\delta du}{dx}+\frac{dw}{dx}\frac{\delta du}{dz}\right)\alpha^2\gamma^2+\\ \left(\frac{du}{dx}\frac{\delta dv}{dy}+\frac{du}{dy}\frac{\delta dv}{dx}\right)\alpha^2\beta^2+\\ \left(\frac{dv}{dx}\frac{\delta dv}{dx}\alpha^2\beta^2+\frac{dv}{dy}\frac{\delta dv}{dy}\beta^4+\frac{dv}{dz}\frac{\delta dv}{dz}\beta^2\gamma^2\right)+\\ \left(\frac{dw}{dy}\frac{\delta dv}{dz}+\frac{dw}{dz}\frac{\delta dv}{dy}\right)\beta^2\gamma^2+\\ \left(\frac{du}{dx}\frac{\delta dw}{dz}+\frac{du}{dz}\frac{\delta dw}{dx}\right)\alpha^2\gamma^2+\\ \left(\frac{dv}{dy}\frac{\delta dw}{dz}+\frac{dv}{dz}\frac{\delta dw}{dy}\right)\beta^2\gamma^2+\\ \left(\frac{dw}{dx}\frac{\delta dw}{dx}\alpha^2\gamma^2+\frac{dw}{dy}\frac{\delta dw}{dy}\beta^2\gamma^2+\frac{dw}{dz}\frac{\delta dw}{dz}\gamma^4\right)\end{cases}$$

where ψ is the angle of the rayon ρ with its projection on the $\alpha\beta$ plane and φ is the angle which this projection forms with the α axis, and then

$$\alpha=\rho\cos\psi\cos\varphi,\quad \beta=\rho\cos\psi\sin\varphi,\quad \gamma=\rho\sin\psi \tag{12}$$

We integrate with respect to φ, ψ from 0 to $\frac{\pi}{2}$ and with respect to ρ from 0 to ∞. By the formulae of the original function on infinite integral :

$$\begin{cases}\int\sin^2 xdx=\frac{1}{2}x-\frac{1}{4}\sin 2x,\quad \int\cos^2 xdx=\frac{1}{2}x+\frac{1}{4}\sin 2x,\\ \int\sin^3 xdx=-\frac{1}{3}\cos x(\sin^2 x+2),\quad \int\cos^3 xdx=\frac{1}{3}\sin x(\cos^2 x+2),\\ \int\sin^n xdx=-\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int\sin^{n-2}xdx,\quad \int\cos^n xdx=\frac{1}{n}\cos^{n-1}\sin x+\frac{n-1}{n}\int\cos^{n-2}xdx,\\ \int\sin x\cos^m xdx=-\frac{\cos^{m+1}x}{m+1},\quad \int\sin^m x\cos dx=\frac{\sin^{m+1}x}{m+1}\end{cases}$$

Excepting for ρ,

$$\alpha^4\cos\psi=\cos^5\psi\cos^4\varphi=\frac{\pi}{10},\quad \beta^4\cos\psi=\cos^5\psi\sin^4\varphi=\frac{\pi}{10},\quad \gamma^4\cos\psi=\cos\psi\sin^4\varphi=\frac{\pi}{10}$$

$$\begin{cases}\alpha^2\beta^2\cos\psi=\cos^5\psi\sin^2\varphi\cos^2\varphi=\frac{\pi}{30},\\ \alpha^2\gamma^2\cos\psi=\cos^3\psi\sin^2\varphi\cos^2\varphi=\frac{\pi}{30},\\ \beta^2\gamma^2\cos\psi=\cos^3\psi\sin^2\varphi\sin^2\varphi=\frac{\pi}{30}\end{cases}$$

Total of the sphere is multiplied by 8 taking ε as the common factor :[9]

$$\varepsilon = \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) \tag{13}$$

We get now ε of (5), and using (4) it turns out the term of $\frac{\varepsilon}{\rho}\Delta\mathbf{v}$ of the today's formulation (10).

$$\varepsilon\begin{cases} 3\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+2\frac{dudv}{dxdy}+2\frac{dwdu}{dzdx}, \\ \frac{d^2v}{dx^2}+3\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+2\frac{dudv}{dxdy}+2\frac{dudw}{dydz}, \\ \frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+3\frac{d^2w}{dz^2}+2\frac{dvdw}{dydz}+2\frac{dwdu}{dzdx} \end{cases}$$

2.2.3. *Navier's deduction of the expression of the summary moments of the forces caused by the reciprocal actions of the molecules of a fluid.*

Navier uses the above results to seek the expression of the summary moments of the forces caused by the reciprocal actions of the molecules of a fluidas follows :

$$\alpha' = \rho\cos\psi\cos\varphi, \quad \beta' = \rho\cos\psi\sin\varphi, \quad \gamma' = \rho\sin\psi$$

We calculate $d\rho d\psi d\varphi\rho^2 cos\psi$ and it turns out as follows :

$$\begin{cases} \alpha'^2\cos\psi = \iint d\psi d\varphi\cos^3\psi\cos^2\varphi, \\ \beta'^2\cos\psi = \iint d\psi d\varphi\cos^3\psi\sin^2\varphi, \\ \gamma'^2\cos\psi = \iint d\psi d\varphi\sin^2\psi\cos^2\psi \end{cases}$$

$F(\rho)$ is the same as $f(\rho)$ in above section, which is a function which depends on the distance ρ between M and M'. We integlate φ and ψ from 0 to $\frac{\pi}{2}$, considering the common value as $\frac{\pi}{6}$, then we get

$$\frac{4\pi}{6}\int_0^\infty d\rho\rho^2 F(\rho) \equiv E$$

We define

$$E(u\delta u + v\delta v + w\delta w)$$

for the expression which we seek for the summary of the moments of the total actions which caused between the molecules of the boudary and the fluid, following the direction which pass by the point of the separation of the fluid and the boundary. E is the constant which will be given by the experiment, due to the nature of the boundary and the fluid, and which can be regarded as the measure of its reciprocal action. Here, we rotate the rectanglar coordinates for γ' to coincide with the direction MN of which M is the common

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[9] O.Darrigol[15, p.112] interprets that this is Navier's tensor as follows :

$$\frac{1}{2}\varepsilon = \frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho) \equiv k, \quad M = \int \sigma_{ij}\partial_i w_j d\tau,$$

$$\sigma_{ij} = -kN^2(\delta_{ij}\partial_k u_k + \partial_i u_j + \partial_j u_i) \equiv -kN^2(\delta_{ij}u_{kk} + u_{ji} + u_{ij}), \quad \text{where } N = 1.$$

In analogy with Lagrange's reasoning, Navier then integrated by parts to get

$$M = \oint \sigma_{ij}\partial_i w_j dS_i - \int (\partial_i\sigma_{ij}) w_j d\tau.$$

origin of the both rectanglar coodinates of α, β, γ and α', β', γ' satisfying $\varphi = r$ and $\psi = s$ and then we get the new relation of α, β, γ from (12).

$$\begin{cases} \alpha = -\alpha' \sin r + \beta' \cos r \sin s + \gamma' \cos r \cos s, \\ \beta = \quad \alpha' \cos r + \beta' \sin r \sin s + \gamma' \sin r \cos s, \\ \gamma = \qquad\qquad \beta' \cos s \qquad\qquad - \gamma' \sin s \end{cases}$$

where each last terms of the right hand-side of α, β, γ are the original values and the rest terms are added by the rotation.

$$\begin{aligned} & F(\rho) V \delta V = \\ \frac{F(\rho)}{\rho} \quad & \{ \quad \alpha'(-u \sin r + v \cos r) + \beta'(u \cos r \sin s + v \sin r \sin s + w \cos s) \\ & + \quad \gamma'(u \cos r \cos s + v \sin r \cos s - w \sin s) \quad \} \\ & \{ \quad \alpha'(-\delta u \sin r + \delta v \cos r) + \beta'(\delta u \cos r \sin s + \delta v \sin r \sin s + \delta w \cos s) \\ & + \quad \gamma'(\delta u \cos r \cos s + \delta v \sin r \cos s - \delta w \sin s) \quad \} \end{aligned}$$

$$\begin{aligned} 0 = & \iiint dxdydz \begin{cases} \left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\right]\delta u \\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\right]\delta v \\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\right]\delta w \end{cases} \\ - & \varepsilon \iiint dxdydz \begin{cases} 3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz} + \frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy} + \frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz} \\ \frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx} + \frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz} + \frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy} \\ \frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx} + \frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy} + \frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz} \end{cases} \\ + & \mathbf{S} ds^2 E(u\delta u + v\delta v + w\delta w). \end{aligned} \tag{14}$$

Here, **S** means the integration in the total surface of the fluid, in varing the quantity E, following the nature of the body with which this surface is contact. Shifting d to the front of δ of the middle term of the right hand-side of (14) and by Taylor expansion using the partial integral

$$\varepsilon \iiint dxdydz \begin{cases} \left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dzdx}\right)\delta u \\ \left(2\frac{d^2u}{dxdy} + \frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2w}{dydz}\right)\delta v \\ \left(2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2}\right)\delta w \end{cases}$$

$$+ \quad \varepsilon \iint dy'dz'\left[\left(3\frac{du'}{dx'} + \frac{dv'}{dy'} + \frac{dw'}{dz'}\right)\delta u' + \left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta v' + \left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta w'\right]$$

$$+ \quad \varepsilon \iint dx'dz'\left[\left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta u' + \left(\frac{du'}{dx'} + 3\frac{dv'}{dy'} + \frac{dw'}{dz'}\right)\delta v' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta w'\right]$$

$$+ \quad \varepsilon \iint dx'dy'\left[\left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta u' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta v' + \left(\frac{du'}{dx'} + \frac{dv'}{dy'} + 3\frac{dw'}{dz'}\right)\delta w'\right]$$

$$- \quad \varepsilon \iint dy''dz''\left[\left(3\frac{du''}{dx''} + \frac{dv''}{dy''} + \frac{dw''}{dz''}\right)\delta u'' + \left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta v'' + \left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta w''\right]$$

$$- \quad \varepsilon \iint dx''dz''\left[\left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta u'' + \left(\frac{du''}{dx''} + 3\frac{dv''}{dy''} + \frac{dw''}{dz''}\right)\delta v'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta w''\right]$$

$$- \quad \varepsilon \iint dx''dy''\left[\left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta u'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta v'' + \left(\frac{du''}{dx''} + \frac{dv''}{dy''} + 3\frac{dw''}{dz''}\right)\delta w''\right]$$

From (8) we get the short expression as follows :

$$\varepsilon \iiint dxdydz\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)\delta u + \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)\delta v + \left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)\delta w]$$

$$+ \quad \varepsilon \iint dy'dz'\left[2\frac{du'}{dx'}\delta u' + \left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta v' + \left(\frac{du'}{dz'} + \frac{dw'}{dx'}\right)\delta w'\right]$$

$$+ \quad \varepsilon \iint dx'dz'\left[\left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right)\delta u' + 2\frac{dv'}{dy'}\delta v' + \left(\frac{du'}{dz'} + \frac{dw'}{dy'}\right)\delta w'\right]$$

$$+ \quad \varepsilon \iint dx'dy'\left[\left(\frac{du'}{dz'} + \frac{dv'}{dx'}\right)\delta u' + \left(\frac{dv'}{dz'} + \frac{dw'}{dy'}\right)\delta v' + 2\frac{dw'}{dz'}\delta w'\right]$$

$$- \quad \varepsilon \iint dy''dz''\left[2\frac{du''}{dx''}\delta u'' + \left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta v'' + \left(\frac{du''}{dz''} + \frac{dw''}{dx''}\right)\delta w''\right]$$

$$- \quad \varepsilon \iint dx''dz''\left[\left(\frac{du''}{dy''} + \frac{dv''}{dx''}\right)\delta u'' + 2\frac{dv''}{dy''}\delta v'' + \left(\frac{du''}{dz''} + \frac{dw''}{dy''}\right)\delta w''\right]$$

$$- \quad \varepsilon \iint dx''dy''\left[\left(\frac{du''}{dz''} + \frac{dv''}{dx''}\right)\delta u'' + \left(\frac{dv''}{dz''} + \frac{dw''}{dy''}\right)\delta v'' + 2\frac{dw''}{dz''}\delta w''\right]$$

We get from (14) as follows :

$$0 = \iiint dxdydz \begin{cases} \left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) - \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)\right]\delta u \\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) - \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)\right]\delta v \\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) - \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)\right]\delta w \end{cases} \quad (15)$$

We get (9) from (15).

By the way, F.Graber, a mathematical historian investigates the differences between [51] and [52] in his paper [21], however, he does not cite precisely this big differences of the equations between (5) and (9) with the mathematical expressions.

2.3. **Poisson's equations.** S.D.Poisson[60, p.151-152](1829-31) proposed his equations in 1829 and was issured in 1831, which reads :

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ \frac{d^2y}{dt^2} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \end{cases}$$

$$(9) \quad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \varepsilon(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \varepsilon(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2}) = 0, \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \varepsilon(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0, \\ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \end{cases}$$

If we put $-\frac{\varepsilon}{\rho} \equiv \nu$ and $\mathbf{f} = (X, Y, Z)$, then Poisson's equations are equivalent with Navier's equations of the incompressible fluid :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}.$$

Poisson contains both compressible and incompressible fluid.

2.4. **Stokes' principle and equation.** Stokes says in [65, p.80] :

> If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle :
>
> - That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P ; and
> - that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.

Stokes proposes the Stokes' approximate equations in [65, p.93] :

$$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0, \\ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \end{cases}$$

> These equations are applicable to the determination of the motion of water in pipes and canala, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

By the modern vectorial expression, if we take $\mathbf{f} = (X, Y, Z)$, then these equations are turn out as follows :

$$\rho(\frac{D\mathbf{u}}{Dt} - \mathbf{f}) + \nabla p - \mu\Delta\mathbf{u} = 0, \quad \text{div } \mathbf{u} = 0.$$

TABLE 4. No.2 Successions of the invariant tensor : t_{ij}

no	name	paper	application	tensor	succession
1	A.Cauchy	[6](1828)	optical into the elastic theory of light	stress tensor	
2	S.D.Poisson	[60](1829-31)	hydrodynamic tensor	stress tensor	
3	G.Green	[22](1828),[23](1839)	theories of electricity and magnetism	Green tensor	1
4	G.G.Stokes	[65](1877)			6,7,8,9,10
5	A.M.Lyapunov	[50](1898)	electrostatic problem	Green tensor	3
6	C.W.Oseen	[59](1927)	hydrodynamic tensor	Green tensor	3
7	L.Lichtenstein	[45](1928),[46]('29),[47]('31)		Green tensor	P.Levi(1919)
8	F.K.G.Odqvist	[57](1930),[58]('32)	hydrodynamic Green tensor	stress tensor	6,7
9	J.Leray	[42](1933),[43, 44]('34)	hydrodynamic tensor	Green tensor	6,8
10	O.A.Ladyzhenskaya	[36](1959),[37](1970)	hydrodynamic tensor	Green tensor	6,7,8,9
11	V.A.Solonnikov	[64](1977)			6,7,8,9,10

2.5. **The naming of the Navier-Stokes equations.** E.Hopf[25] seems to name first what we call the equations, the "*die Navier-Stokes Gleihungen*" ("*the Navier-Stokes equations*"). J.Leray[42, 43, 44] calls consistently them "*des equations de Navier*" ("*the equations of Navier*").

3. THE SUCCESSIONS OF THE FUNDAMENTAL SOLUTIONS

3.1. **The invariant tensor in the Navier-Stokes equations.** We consider the Navier-Stokes equations of the incompressible, homogeneous fluid. At first, we introduce Lagrange derivative :

$$\frac{DF}{Dt}=\frac{\partial F}{\partial t}+\sum_{j=1}^{3}u_j\frac{\partial F}{\partial x_j},\quad \frac{D}{Dt}=\frac{\partial}{\partial t}+\sum_{j=1}^{3}u_j\frac{\partial}{\partial x_j}=\frac{\partial}{\partial t}+u\cdot\nabla=\frac{\partial}{\partial t}+u\cdot grad$$

The responding force on the closed surface S, using the normal vector at the point Q : $n=(n_1,n_2,n_3)$:

$$T_n=T(u,p)\cdot n,\quad T(u,p)=\{T_{ij}(u,p)\}_{i,j=1,2,3},\quad T_{ij}(u,p)=-p\delta_{ij}+\mu(u_{ji}+u_{ij})$$

Here T_{ij} is the invariant tensor. From Gauss' divergence theorem

$$-\iiint_D\frac{Du}{Dt}\rho dx+\iint_S T(u,p)\cdot ndS=0$$

From Gauss' divergence theorem in regarding with $div\ \mathbf{u}=0$

$$\iint_S\sum_{j=1}^{3}T_{ij}(u,p)n_jdS=\iiint_D\sum_{j=1}^{3}\frac{\partial}{\partial x_j}T_{ij}(u,p)dx=\iiint_D\Big(\mu\sum_{j=1}^{3}\frac{\partial^2u_i}{\partial x_j^2}-\frac{\partial p}{\partial x_i}\Big)dx,\quad i=1,2,3$$

$$\iint_S T(u,p)\cdot ndS=\iiint_D(\mu\Delta\mathbf{u}-\nabla p)dx$$

$$\frac{Du}{Dt}=\frac{\partial u}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=-\frac{1}{\rho}\nabla p+\nu\Delta\mathbf{u},\quad where\ \nu=\frac{\mu}{\rho}$$

Here we get (7), assuming the external force $\mathbf{f}=0$.

Navier deduces the expression in using the rectanglar coordinates, but not by the tensor. As O.Darrigol[15, 16] cites as in the above footnote about ε in (13), we can interpret the tensor from Navier's expressions.

TABLE 5. No.3-1 Expression for hydrodynamics by the invariant tensor : t_{ij}

no	name&reference source of t_{ij}	problem
1	A.Cauchy[6, p.225](1828)	
2	S.D.Poisson[60, p.140](1829-31)	
3	G.G.Stokes[65, pp.90-91](1845)	the Stokes equations
4	C.W.Oseen[59, p.26](1927)	the Stokes differential equations
5	K.G.Odqvist[57, p.332](1930)	first Stokes boundary problem: $\mu\Delta u_i = \frac{\partial p}{\partial x}, \ \frac{\partial u_k}{\partial x_k} = 0$
6	J.Leray[42, p.22,p.55](1933)	(25) $\mu\Delta u_i - \frac{\partial p}{\partial x_i} = -\rho X_i \ (i = 1, 2, 3), \ \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} = 0$
7	O.A.Ladyzhenskaya[36](1959)	linearized N-S system:$\Delta \mathbf{u} - \text{grad}p = \mathbf{f}, \text{div } \mathbf{u} = 0, \mathbf{u}\|_S = 0$
8	O.A.Ladyzhenskaya[37](1970)	linearized N-S system: $\nu\Delta \mathbf{u}^k(x, y) - \text{grad}q^k(x, y) = \delta(x - y)\mathbf{e}^k, \text{div } \mathbf{u}^k = 0$
9	V.A.Solonnikov[64, p.48](1977)	$\frac{\partial G_{ij}}{\partial t} - \Delta G_{ij} + \frac{\partial P_j}{\partial x_i} = 0, \quad \sum_{i=1}^{3} \frac{\partial G_{ij}}{\partial x_i} = 0,$ *where* $G_{ij}\|_{x_3=0} = \delta_{ij}\delta(t)\delta(x_1)\delta(x_2)$

TABLE 6. No.3-2 Expression for hydrodynamics by the invariant tensor : t_{ij}

no	definition , where v_i is the velocity vector,$r^2 = (x_j - x_j^{(0)})^2, r > 0$
1	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
2	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
3	$t_{ij} = p\delta_{ij} + (\delta - 2\mu(v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i})$
4	$t_{jk} = \frac{\delta_{jk}}{r} + \frac{(x_j - x_j^{(0)})(x_k - x_k^{(0)})}{r^3}, \quad p_k = -2\mu\frac{\partial}{\partial x_k}\frac{1}{r} = 2\mu\frac{(x_k - x_k^{(0)})}{r^3}$
5	$T_{ik} = -p\delta_{ik} + \mu(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}) \equiv -p\delta_{ik} + \mu(u_{ki} + u_{ik})$
6	$T_{ij}(x, y) = \frac{1}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{(y_i - x_i)(y_j - x_j)}{r^3}\right], \quad P_i(x, y) = \frac{1}{4\pi}\frac{y_i - x_i}{r^3}$
7	$T_{ij}(x, y) = \frac{1}{8\pi}\left[\frac{\delta_i^j}{\|x-y\|} + \frac{(y_i - x_i)(y_j - x_j)}{\|x-y\|^3}\right], \quad P_i(x, y) = \frac{1}{4\pi}\frac{y_i - x_i}{\|y-x\|^3}$
8	$T_{ik}(\mathbf{v}) = -\delta_i^k p + \nu(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i})$ (cite from [57])
9	$G_{ij} = -2\frac{\partial \Gamma}{\partial x_3}\delta_{ij} - \frac{1}{\pi}\frac{\partial C_i}{\partial x_j} - \frac{1}{2\pi}\frac{\partial}{\partial x_i}\frac{1}{\|x\|}\delta_{j3}\delta(t),$ $P_j = \frac{\delta(t)}{2\pi}\frac{\partial^2}{\partial x_j \partial x_3}\frac{1}{\|x\|} + \frac{1}{2\pi t}\frac{\partial A}{\partial x_i} - \frac{\delta'(t)}{2\pi}\delta_{j3}\frac{1}{\|x\|}$

3.2. **Successions of the invariant tensor.** We show the tensors of Cauchy[6, 7, 8, 9, 10, 11, 12], Poisson[60] and Stokes[65] in Table 4,5 and 6. We show here the construction of the tensor, for an example, by C.W.Oseen[59][10].

3.3. **Fundamental solutions for the condition on the velocity components.** [11]

We turn back to our problem, to determine the fundamental solutions of the Stokes differential equations. We said that we shall select these fundamental solutions so that the detail functions v depend only two point P and $P^{(0)}$, moreover, that the system of these functions in all themselves way of the coordinate depended, we also select the right hand direction system. It is easy to assume that these new functions of the components of the one than the transformation (10)[12] of the invariant tensor with the range there are 2. We have used from these underlying, deduced, a tensor which in an arbitrary right hand direction system of the following components : $t_{jk} = \delta_{jk}\Delta\Phi(r) - \frac{\partial^2\Phi(r)}{\partial x_j x_k}, \quad r^2 = (x_j - x_j^{(0)})^2, \quad r > 0 \cdots (11)$, where δ_{jk} is here and bellow the jk-component of a tensor, these diagonal components of

[10]The following "We" is of course the author of [59].

[11]This English version of C.W.Oseen[59] was translated from German by Shigeru Masuda. We use CbT (: Comment by Translator) in pages 1ff to avoid the confusion between the original comment and ours one.

[12]CbT : $x_j' = a_j + l_{jk}x_k, \quad x_j^{(0)'} = a_j + l_{jk}x_k^{(0)}$.

$(j=k)$ have the value 1 and the else components have the value 0. The three functions t_{1k}, t_{2k}, t_{3k} satisfy always, i.e., when k have the value $1,2,3$, the equation : $\frac{\partial t_{jk}}{\partial x_j} = 0 \cdots (12)$. When we define that Φ of the equation : $\Delta_x\Delta_x\Phi = 0, \quad \left(\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \cdots (13)$ should be satisfied and when we put : $-\mu\frac{\partial}{\partial x_k}\Delta_x\Phi = p_k$, so we have for all admitted j- and k-value : $\mu\Delta_x t_{jk} - \frac{\partial p_k}{\partial x_j} = 0 \cdots (14)$. For these k-value ($1,2$ and 3) and also the three functions t_{1k}, t_{2k}, t_{3k} and p_k are one solution of the Stokes equation i.e., we can put the equation (13), Φ depends only on r, to the familia transformation of Δ in the polar coodinate in the form : $\frac{d^4(r\Phi)}{dr^4} = 0$. These generalized solution is also $\Phi = ar^2 + br + c + \frac{d}{r}$, where a,b,c,d are the constants. We put from the basis, which we define soon, $\Phi(r) = r$. We have then :

$$\Delta\Phi = \frac{2}{r}, \quad t_{jk} = \frac{\delta_{jk}}{r} + \frac{(x_j - x_j^{(0)})(x_k - x_k^{(0)})}{r^3}, \quad p_k = -2\mu\frac{\partial}{\partial x_k}\frac{1}{r} = 2\mu\frac{(x_k - x_k^{(0)})}{r^3}.$$

We observe now a domain B of (x_1,x_2,x_3)-space. F is its boundary surface. We assume that the Stokes differential equation has a regular solution in B. We show with $P^{(0)} = (x_1^0, x_2^{(0)}, x_3^{(0)})$ of an arbitrary point in the interior of B. We surround with a sphere with $r=\varepsilon$ and select ε so small that this sphere lies in the interior of the F. $B(\varepsilon)$ is a subspace of B, which includes the exterior of the sphere with $r=\varepsilon$. We use the formula (2)[13] on the domain $B(\varepsilon)$, and we put with $v_{jk} = t_{jk},\ \bar{p} = p_k$. The boundary is consist of the two subspaces of F and the sphere with $r=\varepsilon$. Because the value of rt_{jk} is over even in the point of $P^{(0)}$ is stable and because the boundary of the sphere with $r=\varepsilon$ is propotional with ε^2, we have : $\lim_{\varepsilon\to 0}\int_{r=\varepsilon} t_{jk}\Big(\mu\frac{du_j}{dn} - pn_j\Big)dS = 0$. Moreover $\lim_{\varepsilon\to 0}\int_{r=\varepsilon} u_j\Big(\mu\frac{dt_{jk}}{dn} - p_k n_j\Big)dS = \lim_{\varepsilon\to 0}\int_{r=\varepsilon}\Big\{u_k + 3u_j(x_j - x_j^{(0)})\frac{x_k - x_k^{(0)}}{r^2}\Big\}\frac{dS}{r^2}$. We put : $u_j = u_j^{(0)} + r\varphi, \quad$ where $u_j^{(0)} = u_j(P^{(0)})$, and because we put with φ as a bounded function of the point P in the neighborhood of $P^{(0)}$. We have then because

$$\int_{r=\varepsilon}\frac{dS}{r^2} = 4\pi, \quad \int_{r=\varepsilon}\frac{(x_j - x^{(0)})(x_k - x_k^{(0)})}{r^2}\frac{dS}{r^2} = \frac{4\pi}{3}\delta_{jk},$$

$$\lim_{\varepsilon\to 0}\int_{r=\varepsilon} u_j\Big(\mu\frac{dt_{jk}}{dn} - p_k n_j\Big)dS = 8\pi\mu u_k(P^{(0)}).$$

Therefore :

$$u_k(P^{(0)}) = \frac{1}{8\pi\mu}\int_F\Big\{t_{jk}\Big(\mu\frac{du_j}{dn} - pn_j\Big) - u_j\Big(\mu\frac{dt_{jk}}{dn} - p_k n_j\Big)\Big\}dS. \tag{15}$$

When we get the 12 functions T_{jk} and P_k, which in the interior of the boundary F, we can put in the form of : $T_{jk} = t_{jk} + \tau_{jk}, P_k = p_k + \tau_k$, where τ_{jk}, p_k for all k-value ($k = 1,2$ or 3) of the interior of F, the regular solution of the given Stokes equations, so we can deduce directly owing to the product of (15) t_{jk}, p_k by T_{jk}, P_k. If the new T_{jk} all vanish when the point

[13]CbT : $\int_F \Big|v_j\Big(\mu\frac{du_j}{dn} - pn_j\Big) - u_j\Big(\mu\frac{dv_j}{dn} - \bar{p}n_j\Big)\Big|dS = 0. \quad (2)$

P enters onto the boundary F, so we get : $u_k(P^{(0)}) = -\frac{1}{8\pi\mu}\int_F u_j\Big(\mu\frac{dT_{jk}}{dn} - P_k n_j\Big)dS \cdots (16)$.

4. The succession of the weak solutions

4.1. Leray's introduction to construct solution turbulente. Leray[44, p.195] says : [14]

> If I should succeed to construct the solution of the equations of Navier which become irregular, I shall have the right to insist that there exist effectivelly the solutions turbulentes marely no reducing, in the solutions regulieres. Similarly, if this proposition should be false, the notion of solution turbulente which will play no role any longer in the study of viscous liquid, will do no harm to its interest : it must well present the problems of mathematical physics for which physical cause of regularity is not sufficient to justify the hypothesis of the regularity made in setting of equation. To this problem we can then apply the considerable resemblance to that which I expose here.

4.2. Leray and Hopf. E.Hopf[28] comments on his own lemma 5.1 to the J.Leray[44] in [28] : [15]

> In the Rellich's theorem, the convergence of the x-integral on the quadratic of the derivation is presupposed. Our convergence presupposition relate even to the (x,t)-integral and is therefore better adapted to the situation in our problem. Leray prove and use **Lemma** 2, which is even near to Rellich's lemma, operate like this theorem, only with (x)-integral. Our proof of convergence is more direct.

5. The succession of the generalized solution / the strong solution

5.1. Sobolev embedding theorem. We see Sobolev[63] owes to the potential theory from Weyl[69][16]. [cf. Sobolev[63, pp.42-54], §1.7 (The spaces of $L_p^{(l)}$ and $W_p^{(l)}$).] In his contents, he deduces his embedding theorem explaining by the same as Oseen's tensor[59] with following his decomposition :

> We consider also the space S_l of all polynomials of degree at most $l-1$. The so-defined norm will be invariant under all rotations of axes of cordinates, while in distinction from the norm on L_p^l it will no longer be invariant under the translation of the origin. Indeed the quality $\sum k!/(\alpha_1!\alpha_2!\cdots\alpha_n!)a^2_{\alpha_1\alpha_2\cdots\alpha_n}$ is one of the invariant of the tensor $a_{\alpha_1\alpha_2\cdots\alpha_n}$ and therefore this quality is preserved under orthgonal transformations. $\|\varphi\|^p_{W_p^{(l)}} = \|\Pi_1\varphi\|^p_{S_l} + \|\Pi_1^*\varphi\|^p_{L_p^{(l)}} = \|\Pi_1\varphi\|^p_{S_l} + \|\varphi\|^p_{L_p^{(l)}} \cdots (7.5)$. The equation (9) in [34] [17] is deduced from the equations : (7.4), $\cdots$, (7.15) after substituting with $n=3, p=2$ and

[14]This English version from French was made by Shigeru Masuda.

[15]This English version from Germany was made by Shigeru Masuda.

[16]In [63], Sobolev does not mention on Oseen's tensor [59] but Weyl[69], and moreover Ladyzhenskaya[?] does not mention on Sobolev[63] but Weyl[69] in the references of one of the last papers in her life. She owes more to [69].

[17]The equation (9) of Kiselev & Ladyzhenskaya[34] is as follows : $u(x) = \frac{1}{\chi}\int_{\Omega_\delta} u(y)v(y)dy - \frac{1}{\chi}\int_{\Omega_\delta}\Big[\frac{1}{|x-y|^2}\frac{y_k-x_k}{|x-y|}\varphi_{y_k}(y)\int_{|x-y|}^{\infty} v\Big(x+\rho\frac{x-y}{|x-y|}\Big)\rho^2 d\rho\Big]dy \equiv c_u + \frac{1}{\chi}\int_{\Omega_\delta}\frac{1}{r^2}\omega_k(x,y)\varphi_{y_k}(y)dy \equiv c_u + \tilde{u}(x) \quad (9)$.

$l=1$(i.e. W_2^1) or 2(i.e. W_2^2 in Theorem 4 in §5 in Kiselev & Ladyzhenskaya's paper[34]). We refer below only (7.14) and (7.15) in [34].

$$\begin{aligned}\varphi(\vec{P}) &\\ = & \sum_{\sum \alpha_i \le l-1} (x_1-y_1)^{\alpha_1}\cdots(x_n-y_n)^{\alpha_n}\int_\Omega \zeta_{\alpha_1\cdots\alpha_n}(\vec{Q}-\vec{T})\varphi(\vec{Q})dv_{\vec{Q}} \\ + & \int_\Omega \frac{1}{r^{n-1}}\sum_{\sum\alpha_i=l} w_{\alpha_1\cdots\alpha_n}(\vec{Q}-\vec{T},\vec{P}-\vec{T})\frac{\partial^l\varphi(\vec{Q})}{\partial y_1^{\alpha_1}\partial y_2^{\alpha_2}\cdots\partial y_n^{\alpha_n}}dv_{\vec{Q}}. \quad (7.14)\end{aligned}$$

If the point $\vec{T}$ is the origin, then (7.14) passes into (7.12). Further, we set $\tilde{\varphi}(\vec{P})=S(\vec{P})+\varphi^*(\vec{P})\cdots(7.15)$. Thus, for functions φ having continuous derivatives up to order l, φ coincides with $\tilde{\varphi}$ on Ω, and for functions $\varphi\in W_p^{(l)}$ it is equal to $\tilde{\varphi}$ almost everywhere on Ω.

5.2. **Kiselev.** Kiselev[33] is one of the pioneer of the **generalised solution** and the **strong solution** as follows :

$L\mathbf{v}\equiv\frac{\partial \mathbf{v}}{\partial t}+\sum_{k=1}^3 v_k\frac{\partial \mathbf{v}}{\partial x_k}-\nu\Delta\mathbf{v}=-\text{grad } p+\mathbf{f}\cdots(1)$, div $\mathbf{v}=0\cdots(2)$, $\mathbf{v}|_{t=0}=\mathbf{a}\cdots(3)$, $\mathbf{v}|_S=0\cdots(4)$, where $\mathbf{f}=\mathbf{f}(x,t)$ and $\mathbf{a}(x)$ is the given vector, ν is the viscosity coefficient, which, for the brief description's sake, (we) deal as the constant. (We) call the vector $\mathbf{v}$ the **generalised solution** of the problem (1)-(4) on Q_t, if $\mathbf{v}\in L^2(Q_t)$, exists generally in the sense of S.L.Sobolev[63].

Theorem 1 (**Uniqueness theorem**). *The problem(1)-(4) have in Q_t not more than a generalised solution*

Theorem 2 (**Existence theorem 1**). *Supposing $\mathbf{a}\in W_2^{(2)}$ and satisfies the conditions (2) and (4), $\mathbf{f}\in L_2(Q_t)$ and $\frac{\partial \mathbf{f}}{\partial t}\in L_2(Q_t)$ and satisfies the condition $\|\mathbf{a}\|\ \{\ \|\mathbf{f}+L\mathbf{a}\|+\|\mathbf{f}\|\ \}_{t=0}<\frac{\nu^3}{\beta^2}$ where β : a constant, depending on the domain Ω, and the symbol $\|\cdot\|$ means the norm in $L_2(\Omega)$. Then the problem (1)-(4) have the generalised solution, in any cases, for all $t\in[0,T]$, where T : an arbitrary number $\le l$, satisfying $\left(\|\mathbf{a}\|+\int_0^T\|\mathbf{f}\|dt\right)\left(\|\mathbf{f}-L\mathbf{a}\|_{t=0}+\max_{0\le t\le T}\|\mathbf{f}\|+\int_0^T\|\frac{\partial \mathbf{f}}{\partial t}\|dt\right)<\frac{\nu^3}{\beta^2}$.* □

5.3. **Kiselev and Ladyzhenskaya.** They say in [34]:

the difficulties to seak for the **classical solutions** of the problem (1) and the apprehension of it, which this problem may not have such solutions ≪ in the large ≫ , forced to seak for another ≪ generalized solution situated in the problem (1)≫.

In (our)[18] paper, (we) study the problems of the incompressible viscosity:

$$\frac{\partial \mathbf{v}}{\partial t}-\nu\Delta\mathbf{v}+\sum_{k=1}^3 v_k\frac{\partial \mathbf{v}}{\partial x_k}=-\text{grad } p+\mathbf{f}(x,t),\quad \text{div } \mathbf{v}=0,\quad \mathbf{v}|_S=0,\quad \mathbf{v}|_{t=0}=\mathbf{a}\qquad(1)$$

[18]We refer the original [15] in using (we/our). This English version from Russian was made by Shigeru Masuda. The first English version : Amer. Math. Soc., Transl(2) **24**(1963) by John Abramowich without corrections and comments. After conveying deep gratitude to him, we corrected the original misprints, amended phrases and words.

and the first boundary value problem for the system

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + \sum_{k=1}^{3} v_k \frac{\partial \mathbf{v}}{\partial x_k} = \mathbf{f}(x,t), \quad \mathbf{v}|_S = 0, \quad \mathbf{v}|_{t=0} = \mathbf{a} \tag{2}$$

Formulation 1. (We) shall call it a **generalized solution** of problem (1), that is the vector function $\mathbf{v}(x,t)$, having the generalized derivatives $\in L_2(Q_T)$ of the first order, summing to the power of 4 in a plane of $t =$ const for an arbitrary profile Q_T, $\int_\Omega \sum_i v_i^4(x,t)dx <$ const, and satisfying the conditions: div $\mathbf{v} = 0$, $\mathbf{v}|_S = 0$, $\mathbf{v}|_{t=0} = \mathbf{a}$ and the equality: $\int_0^T \int_\Omega \left[\frac{\partial \mathbf{v}}{\partial t}\mathbf{\Phi} + \nu \frac{\partial \mathbf{v}}{\partial x_k}\frac{\partial \mathbf{\Phi}}{\partial x_k} - v_k \mathbf{v}\frac{\partial \mathbf{\Phi}}{\partial x_k} - \mathbf{f}\mathbf{\Phi}\right] dxdt = 0 \cdots (3)$ $^\forall\mathbf{\Phi} \in L_2(Q_T)$ such that $\frac{\partial \mathbf{\Phi}}{\partial x_k} \in L_2(Q_T)$, div $\mathbf{\Phi} = 0$, $\mathbf{\Phi}|_S = 0$. □

Formulation 2. (We) shall call it a **generalized solution** of problem (1), that is the vector function $\mathbf{v}$(x,t), having the generalized derivatives $\in L_2(Q_T)$ in the form of $\frac{\partial^2 \mathbf{v}}{\partial t \partial x_i}$ and its all belongings satisfing the same condition as in **Formulation 1**.

Formulation 3. (We) shall call it a **generalized solution** of problem (1), that is the vector function $\mathbf{v}$(x,t), having the generalized derivatives $\in L_2(Q_T)$ in the form of $\frac{\partial \mathbf{v}}{\partial x_i}$ satisfying the conditions: div $\mathbf{v} = 0$, $\mathbf{v}|_S = 0$ and the equality : $\int_0^T \int_\Omega \left[\mathbf{v}\frac{\partial \mathbf{\Phi}}{\partial t} - \nu \frac{\partial \mathbf{v}}{\partial x_k}\frac{\partial \mathbf{\Phi}}{\partial x_k} + v_k \mathbf{v}\frac{\partial \mathbf{\Phi}}{\partial x_k} + \mathbf{f}\mathbf{\Phi}\right] dxdt + \int_\Omega \mathbf{a}\mathbf{\Phi}(x,0)dx = 0 \cdots (4)$, $^\forall\mathbf{\Phi} \in W_2^1(Q_T)$ such that div $\mathbf{\Phi} = 0$, $\mathbf{\Phi}|_S = 0$, $\mathbf{\Phi}|_{t=T} = 0$. □

(The following 3 theorems are new contents in [34] in comparison with [32, 33].)

Theorem 3. *The problem(1) and (2) can not have more than unique* **generalized solution** *in the sense of* **Formulation 1** *and moreover,* **Formulation 2**. □

Theorem 4. *Problem(1) and (2) can not have more than unique* **generalized solution** *in the sense of* **Formulation 3**. □

Theorem 5. *If* $\mathbf{a}$ *and* $\mathbf{f}$ *are the continuous functions satisfying Hölder condition for* $\mathbf{x}$ *with such as positive power :* γ*, then the problem (2) has the* **generalized solution** *in the sense of* **Formulation 3**. *Uniqueness has been proved in §2. For this solution, we have the strict inequalities :* $|\mathbf{v}| \le c_1'$, $\int_0^T \sum_k \|\mathbf{v}_{x_k}\|^2 dt \le c_2'$, *here a constants* c_1' *and* c_2' *depend on only the value of* ν, T *and* $\max(|\mathbf{a}|, |\mathbf{f}|)$. □

(The following 2 theorems are same as Kiselev[32, 33]. Theorem 6 is about a strong solution which is already in [33].)

Theorem 6. *If* $\mathbf{a} \in J_{0,1}(\Omega) \cap W_2^2(\Omega)$, *and* $\mathbf{f}$ *and* $\mathbf{f}_t$ *are* $\in L_2(Q_t)$, $Q_t = \Omega \times [0,l]$, *then the problem (1) has the* **generalized solution** *in the sense of* **Formulation 2** *on the cylinder* $Q_T = \Omega \times [0,T]$, *such that* T *: no-smaller than an arbitrary number, depending on* ν, $\|\mathbf{a}\|_{W_2^2(\Omega)}$, $\|\mathbf{f}\|_{L_2(Q_l)}$, $\|\mathbf{f}_t\|_{L_2(Q_l)}$ *and the scale of the domain* Ω.[19]

[19] cf. Kiselev [33, p.27].

Theorem 7. [20] *If* $\mathbf{f} \equiv 0$ *(i.e., the exterior force has a potential) and satisfies the conditions* $\|\mathbf{a}\|\|L\mathbf{a}\| < \frac{\nu^3}{\beta^2}$, $\beta = \sqrt{3}c_\Omega^2$, *then the problem (1) has the unique* **generalized solution** *under an arbitrary* T. *This solution approaches to zero as* $t \to \infty$, *as an element of* $W_2^1(\Omega)$. *The existence of the solution* $^\forall t \geq 0$ *under the conditions of the theorem, follows as we have showed just now from* **Lemma 6** *and* **7**. □

5.4. Ladyzhenskaya. [21]

The motion of the viscous incompressible fluid for the model of Navier-Stokes is described by the four functions : $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x)), andp(x)$, satisfying the equations : $\Delta \mathbf{u} - \text{grad } p = \sum_{k=1}^3 u_k \frac{\partial \mathbf{u}}{\partial x_k} + \mathbf{f}$, (1), $\text{div } \mathbf{u} = 0$, (2), where $f(x) = (f_1(x), f_2(x), f_3(x))$: the vector of the mass force, $\mathbf{u}(x)$: the vector of the velocity of the flow of the fluid at the point of $\mathbf{x} = (x_1, x_2, x_3)$, and $p(x)$: the pressure at the point. For the brief, of the description of the coefficients of the viscosity and the density of the location, we put by regarding as 1. We shall study the motion in this domain Ω of the 3 dimensional, Euclidian space E_3, having its fixed boundary S (S may consist of an arbitrary isolated, closed surface). The case of the moving boundary, we assume the similar investigation. To the boundary S, we assume the essential, incidental condition : $\mathbf{u}|_S = 0$, (3).

Theorem 1. *The problem (8) has* $\mathbf{u}$ *and moreover unique* **generalized solution** *from* H *for any* $\mathbf{f}$, *being the linear functional in* H. □

Theorem 2. *The problem (1)-(3) in the bounded domain* Ω *has, at least, the unique* **generalized solution** *from* H *for any linear functional* $\mathbf{f}$ *in* H, *in paticular, for all* $\mathbf{f}$, *integrable to the power of* $\frac{6}{5}$ *in* Ω. □

Theorem 3. *The problem (1)-(3) with zero satisfy in infinity, have, at least, the unique* **generalized solution** $\in H(\Omega)$ *for the unbounded domain* Ω, *if all* $\mathbf{f}$ *define the linear functional in* $H(\Omega)$ *(the enough conditions of this given in (2)-(3) and Result 1).* □

Theorem 4. *The problem of (1),(2) and (14) have at least, the unique* **generalized solution** *for all* $\mathbf{f}$, *being by the liniar functional in* $H(\Omega)$. □

Theorem 5. *The problem of the sketching the system* n *of the solid, the flow equal in infinity :* $\mathbf{u}_\infty =$ *const , have always, at least, the unique,* **generalized solution** *with respect to all* $\mathbf{f}$, *satisfying the linear functional in* H, *in paticular, with respect to* $\mathbf{f} \equiv 0$. □

Theorem 6. *The problem(1)-(3) have, at least, the unique solution* $u_i(x)$, *continuous together with derivatives in the first order in* Ω *and having continuously differentiable in the second order in the interior of* Ω. *The pressure* $p(x)$ *has a continuous in* Ω *and continuously differentiable in the interior*

[20] cf. Kiselev, [33, p.30],[?, p.874].

[21] This English version from Russian with comments was made by Shigeru Masuda. The first English version : Amer. Math. Soc., Transl(2) 25(1963), 173-197, by Henry Merkelo without corrections and comments. After conveying deep gratitude to him, we corrected the original misprints, amended phrases and words on the translation and added our comments for understand as possible as we can.

of Ω. With respect to $f_i(x)$, we seem such that they satisfy the Hölder's condition with an arbitrary positive constant. □

5.5. **Prodi.** [22] Prodi[61] is maybe one of the inventor with J.L.Lions [48][23] of the modern style conbinating with the function spaces :

when B is a space of Banach, (we) put $u \in L^p(0,\tau;B)$. This means as follows : u is the function of t with the value in B, and integrable to the power of p within the interval : $(0,\tau)$. In special case, $L^p(0,\tau;L^p)$ is equivalent with $L^p(\Omega \times (0,\tau))$. By setting p and q as the number such that $p > 3$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. (We) have evidently $2 < q < 6$. $\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} - \mu \Delta_2 u_j = -\frac{\partial p}{\partial x_j} + f_j$, $\frac{\partial u_j}{\partial x_j} = 0$, $(j = 1,2,3)$.

Theorem 1. *A function u which is a solution of the defined problem is unique if satisfing the following condition $u \in L^{\frac{2p}{p-3}}(0,\tau;L^p(\Omega))$ by the arbitrary value of p, with $3 < p \leq +\infty$.* □

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