確率論におけるノイズの歴史

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1 Noise

Noise is expressed as an idea, a subject, a field, an instrument. Noise came upon the scene with a power and swiftness that transformed all of science and views of the nature of matter. Generally, noise has undesired meaning because of its interference. However, noise is utilized in many fields of science and technology in solving great problems in the world.

Noise came into public in 1905 when Einstein invented aiming at problem of the existence of atom. At that time there was a great debate among the scientists on the topic of existence and non-existence of atoms. There were many non-atomists who did not believe the existence of atoms before Einstein's 1905 paper.

Noise is first discovered by Brown, then mathematical explanation is given by Einstein. Later Langevin made simpler and Perin had confirmed.

2 Brownian motion

Brownian motion was discovered after the invention of Microscope.

Robert Brown

A famous botanist Robert Brown made microscopic observations on the minute particles contained in the pollen of plants, he observed the highly irregular motion of these parts which is now called Brownian motion.

His literatures on Brownian motion are

1. *A Brief account of microscopical observations made in the months of June, July, August, 1827 on the particles contained in Pollen of plants and on the general existence active molecules in organic and inorganic bodies. Philos. Mag., vol. 4, pp. 161-173,1828.*
2. *Additional remarks on active molecules, Philos. mag., vol. 6, pp. 161-166, 1829.*

Albert Einstein

Ultimate method that defines greateness in science is

"Predict an effect, derive a specific formula and let the world perform the experiments".

Einstein followed this method; he said "if atom exists, I predict an effect and derive a specific formula related to the effect".

He wanted to prove the existence of atoms and he invented "Noise" which could be used for the calculation of Avogadro's number.

We should pause here. In 1811, Enter Avogadro argued that in a fixed volume the number of little balls of a gas in the same for all substances, regardless of the mass, the size, or the exact nature of the substance. This was the Avogadro's number which was not recongnized for many years. He had no idea to identify his number. Einstein and more particularly Perrin thought of it seriously. Perrin discussed many different ways to calculate Avogadro's number which was not named at that time. He got the Nobel prize in 1927 after accepting the idea of atoms by everybody.

We now come back to our topic. In 1905, Einstein gave an empirical evidence for the existence of atoms in his paper :

On the Movement of Small Particles Suspended in a Stationary Liquid demanded by the Molecular Kinetic Theory of Heat", Annalen der Physik. 17, p549-560, 1905.

Before this paper, atoms were recognized as a useful concept, but there were many great scientists who did not belie ave the existence of atoms. Einstein's statistical discussion of atomic behavior gave experimentalists a way to count atoms by looking through an ordinary microscope.

Einstein purposed to find a measurable experimental appearence of atoms. He derived the equation of the density $f(x, t)$ of polen grains (the small particle) per unit length at position x at time t as

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}, \tag{2.1}$$

which is a famous diffusion equation and D is the diffusion coefficient.

Taking the initial state at some point, say y, so that $f(x,0) = \delta(x-y)$, then

$$f(x,t) = \frac{1}{\sqrt{4\pi D}}\frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{t}}. \tag{2.2}$$

with the standard deviation

$$\lambda_x = \sqrt{2Dt}, \tag{2.3}$$

which shows that the mean displacement is proportional to the square root of time. He derived the diffusion coefficient in terms of the fundamental constants

$$D = \frac{RT}{N}\frac{1}{6\pi KP}, \tag{2.4}$$

where R is the usual gas constant, T is the temprature, k is the viscosity (now it is called Boltzmann's constant), and P is the radius of the small particle.

Then it is obtained

$$\lambda_x = \sqrt{t}\sqrt{\frac{RT}{N}\frac{1}{3\pi kP}}. \tag{2.5}$$

Solving the above equation for taking a unit time, then

$$N = \frac{1}{\lambda_x^2}\frac{RT}{3\pi kP}. \tag{2.6}$$

N is the Avogadro's number. We can see that it is obtained from fluctuations and noise.

Other literatures written by Einstein on Brownian motion are

1. *On the theory of the Brownian Movement, Annalen der Physik. 19, 1906, 371-381.*
2. *Theoretical Observations on the Brownian motion, 1907*
3. *The elemental theory of the Brownian motion,1908*

Louis Bachelier

Brownian motion was also explained by Louis Bachelier in 1900 in his PhD thesis

"The theory of speculation"

where he discussed option prices, the most interesting time series. He developed many mathematical ideas, which is a part of stochastic processes.

In thesis report, written by the committee, it was emphasized that

The manner in which M. Bachelier deduces Gauss's law is very original and all the more interesting is that his reasoning can be extended with a few changes to the theory of errors.

Bachelier started with the idea of the trasitive property of a probability distribution,

$$P_{z,t_1+t_2} = \int_{-\infty}^{\infty} P_{x,t_1} P_{z-x,t_2} dx.$$

and shows that the solution is the same with the formula (2.2), derived by Einstein five years later.

3 Mathematics of Noise

Mathematics of Noise was invented by Bachelier, Einstein, Smoluchowski, and Langevin.

We have already discussed the contributions of Bachelier and Einstein in the previous section.

Paul Langevin

Langevin was a student of Pierre Curie, a husband of nobel prize winner Marie Curie. Stochastic differential equation was born in Langevin's classical 1908 paper,

On the theory of Brownian motion.

He aimed to obtain the explicit expression for the standard deviation $\lambda_x (= \sqrt{\bar{x}^2})$ for Einstein's equation (2.1), the former is Einstein's notation and the later is his notation. He used Newton's equation :

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X, \tag{3.1}$$

where a is the radius, μ is the viscosity, X the irregular force which we now called is random force. Using the formula

$$m\frac{\overline{dx^2}}{dt} = \frac{RT}{N}, \quad \text{(kinetic energy)}$$

and some tricks he obtained the same value of standard deviation as Einstein's.

Here we note that Einstein's method gives the whole distribution, however Langevin gave only the first and second moment.

Smoluchowski

Smoluchowski had done almost everything what Einstein had done. He worked on Brownian motion from the view points of phisics and mathematics. For instance, the behaviour of the particle where there is an external force such as force of gravity, the concept of a transition probability, among other standard issues. His work was mentioned by Chandrasekhar that the theory of density fluctuations developed by Smoluchowski is one of the most outstanding achievements and is chiefly remmembered as the originator (along with Einstein) of the theory of Brownian motion.

Ornstein, Wang, Uhlenbeck, Furth, Chandrasekhar, Kramers, Rice and others wrote the fundamental mathematical papers in the 50 year period on this subject after the initial developement by Einstein, Bachelier, Smoluchowski and Langevin.

Nobert Wiener

As is well known Nobert Wiener contributed in control theory, filtering, stochastic processes and noise. He defined a purely mathematical view point of Brownian paths in 1923.

Perin pointed out the fact that Brownian paths are continuous, however they are not differentable since they can suddenly change direction. It can be seen in another way that the standard deviation, say σ goes as the square root of time t thus $\frac{\sigma}{t}$ diverges when t tends to zero.

Wiener formulated Brownian motion paths with measure theory. He defined measure for Brownian paths. That is, he defined a stochastic process, from a mathematical view point, as an idealization of Brownian motion formulated by Einstein and others.

The existence of the process can be proved by Wiener's mathematical idealization of Brownian motion.

The solution of the differential equation

$$\frac{dW(t)}{dt} = x(t), \tag{3.2}$$

is called white noise which is a random function such that

$$E(x(t)) = 0, \quad E(x(t)x(s)) = \delta(t-s).$$

That is the process $W(t)$ can be defined as

1. $E(W(t)) = 0.$
2. $E[(W(t) - W(s))^2] = t - s$
3. If $t_1 < t_2 < \cdots < t_n$ then $W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})$ are mutually independent Gaussian random variables.

Most people called this process Wiener process, however the equation (3.2) is a particular case of the Langevin equation and its basic properties have been developed by many many many mathematicians and sicientists before, mostly due to Bachelier, Einstein and Smoluchowski.

In many mathematical literaturs concerned with Brownian motion, the impressive works of the great mathematicians discussed above, are not mentioned (eccept Einstein).

Others

After Wiener started a mathmatical view point of Brownian motion, subsquent contributions were made by Lévy, Kolmogorov, Doob and others.

The literatures to be mentioned here are as follows.

M. C. Wang and G. E. Yhlenbeck, *On the theory of Brownian motion, Phys. Rev., vol 36.833-841, 1936.*

P. Lévy, *Processus stochastiques et mouvement Brownien, Gautier-Villars, Paris, (1948).*

J.L Doob, *The Brownin movement and stochastic equation, Ann. Math., vol 43, p. 351, 1942.*

K. Ito, *Stochastic integrals, Proc. Imp. Acad Tokyo, vol 20. 519-624, 1944.*

Doob reformulated the Langevin equation in a rigourous way. His problem is to find stochastic analog of the Langevin equation. He rewrote the Langevin equation

$$du(t) = -\beta v(t)dt + dx(t).$$

With the aim to give these differentials a suitable intrpretation, he developed the properties of the differentials, which basically means the differences, not in the usual sense of derivatives.

Ito converted the more general type of Langevin equation :

$$\frac{du(t)}{dt} = a(u(t), t) + b(u(t'), t')F(t')dt',$$

to

$$u(t) - u(0) = \int_0^t a(u(t'), t')dt' + \int_0^t b(u(t'), t')F(t')dt'.$$

This is a typical way of writing Langevin equation. The problem of differentiability of stochastic variables is transformed to the problem of defining the integral of a process.

Ito defines stochastic integral by

$$\lim_{N\to\infty} \int_0^t B(t')dF(t')dt' = lim_{N\to\infty} \sum_{i=1}^{N} B(t_{i-1})(F(t_i) - F(t_{i-1})).$$

The Ito type stochastic differential equation has become famous all over the world, not only mathematical circle but widely in the fields of applications.

If we discuss stochastic differential equation ,we should not forget Bernstein's work in 1938 (rather earlier), we refer

S. Bernstein, *Équations Différentilles Stochastiques, Actualités Sci, et Ind. 738(1938) 5-32.*

He actually succeeded in letting ordinary differential equations to be stochastic differential equations. His idea is that, in the stage of approximation, he did put a term of order $\sqrt{\delta t}$ in addition to the term of order dt.

4 White noise analysis (Hida Calculus)

In 1975 Takeyuki Hida introduced white noise theory in his monograph "Analysis of Brownian Functionals" (Carleton Mathematical Lecture Notes no. 13) which has generated a tremendous amount of research during the last quarter century. His ideas and work are truly influential in many ways.

White noise is referred to as a sound with equal intensity at all frequencies within a broad band. The word "white" is used because of its similarity to "white light" which is made up of all different colors (frequencies) of light combined together. In applied science white noise is often taken as a mathematical idealization of phenomena involving sudden and extremely large fluctuations. It is informallydefined as a stochastic process $z(t)$ such that the $z(t)$'s are independent and for each t, $z(t)$ has mean 0 and variance ∞ in the sense that $E(z(t)z(s)) = \delta(t-s)$, where δ is the Dirac delta function. Obviously $z(t)$ is not an ordinary random variable for each t. White noise can also be regarded as the derivative $\dot{B}(t)$ of a Brownian motion $B(t)$. Since $|B(t+h) - B(t)| \sim |h|^{1/2}$ for small h, $\dot{B}(t)$ does not exist in the ordinary sense for each t.

The white noise $\dot{B}(t)$ had long been used in integration before 1975 by engineers like A. V. Balakrishnan. For example, the integration by parts formula can be used to define the informal integral

$$\int_a^b f(t)\dot{B}(t)\,dt$$

as

$$f(t)B(t)]_a^b - \int_a^b f'(t)B(t)\,dt$$

for a C^1-function f. On the other hand, in Itô's theory of stochastic integration it is well known the integral

$$\int_a^b f(t)dB(t)\,dt$$

as the Itô integral for a nonanticipating stochastic process $f(t)$ with almost all sample paths being square integrable. However, a simple integral such as $\int_0^1 B(1)\,dB(t)$ is not defined as an Itô integral. Hida envisioned that this integral should be viewed directly as $\int_0^1 B(1)\dot{B}(t)\,dt$, namely, as a white noise integral.

A few years before 1975 Hida had advocated his ideas of white noise theory in various conferences. What he had in mind was *to introduce a mathematical theory so that* $\dot{B}(t)$ *is meaningful for each* t *and the collection* $\{\dot{B}(t)|\, t \in R\}$ *can be used as a continuum coordinate system.* Moreover, he outlined his vision of complex white noise analysis, random fields, infinite dimensional rotation groups, infinite dimensional harmonic analysis, etc. Looking back more than twenty-five years ago, one cannot help wondering how Hida came up with such a bold attempt. Perhaps it is best described by a remark of Loren Pitt referring to Hida's lecture for the 1975 Multivariate Analysis conference in Pittsburgh "I did not understand all the mathematics in his lecture, but I have the feeling that it is something going to be very important."

It is true that Hida was influenced by Paul Lévy's work on Brownian motion and functional analysis. But his mathematical theory of white noise also stemmed from his own earlier work, in particular, the paper "Canonical representation of Gaussian processes and their applications (Memoirs Coll. Sci., Univ. Kyoto, A. 33, 1960, 109-155) and the book "Stationary Stochastic Processes" (Princeton University Press, 1970.) In the canonical representation of Gaussian processes, known as Hida-Cramér theorem, we can see the root of the white noise differentiation operator. In his Princeton University Press book we find Hida's vision of harmonic analysis on white noise space which eventually led to the concepts of generalized multiple Wiener integrals and generalized Brownian functionals in his Carleton University lecture notes.

Paul Lévy discussed harmonic analysis on the space $L^2(0,1)$ in his book "Leçons d'analyse fonctionnelle" (Gauthier-Villars, 1922.) Consider, for example, a simple function $F(\xi) = \int_0^1 \rho(t)\xi(t)^2\,dt$ defined on $L^2(0,1)$. This function can be regarded as an infinite dimensional analogue of the function $f(x) = \sum_{k=1}^n a_k x_k^2$ on R^n. The Laplacian of f is given by $\Delta f(x) = 2\sum_{k=1}^n a_k$, which suggests us that the corresponding infinite dimensional Laplacian of F should be $\Delta F(\xi) = 2\int_0^1 \rho(t)\,dt$. Hida was inspired by Lévy's idea along this line and tried to interpret nonlinear functions defined on $L^2(0,1)$ and the Lévy Laplacian of such functions from the white noise viewpoint.

Here is Hida's idea to define $\dot{B}(t)$ for each t and nonlinear functons of $\{\dot{B}(t)|t \in R\}$ as generalized Brownian functionals. Let μ be the standard Gaussian measure on the dual space $\mathcal{S}^*$ of the Schwartz space $\mathcal{S}$ on the real line R. The probability space $(\mathcal{S}^*, \mu)$ is called a white noise space since its elements can be regarded as informal sample paths of white noise. Apply the Wiener-Itô theorem to decompose $L^2(\mathcal{S}^*, \mu)$ into an orthogonal direct sum

$$L^2(\mathcal{S}^*, \mu) = \sum_{n=0}^{\infty} \mathcal{H}_n$$

of multiple Wiener integrals or homogeneous chaos.

The limit

$$\dot{B}(t) = \lim_{\epsilon \to 0} \epsilon^{-1}(B(t+\epsilon) - B(t))$$

does *not* exist in the space $\mathcal{H}_1$, as is well known almost all sample functions of a Brownian motion are continuous but not differentiable. Thus if it is smeared the variables by a smooth function ξ:

$$\langle \dot{B}, \xi \rangle = \int \dot{B}(t)\xi(t)dt = -\int B(t)\xi'(t)dt. \tag{4.3}$$

Thus an ordinary Gaussian random variable is obtained. Easy computations prove

$$E(\langle \dot{B}, \xi \rangle \langle \dot{B}, \eta \rangle) = \int \xi(t)\eta(t)dt. \tag{4.4}$$

In this stage, $\langle \dot{B}, \xi \rangle \langle$ can be extended to the case where ξ is in $L^2(R^1)$. However, in order to define $\dot{B}(t)$ itself we have to extend ξ to a class of generalized function including delta function. *Hida's idea* is to extend $L^2(R^1)$ further to $K^{-1}(R^1)$, the Sobolev space of order -1.

Having been given the system of varibles $\{\dot{B}(t)\}$ he established a class of functions, actually functionals.

The most elementary and basic functions, namely polynomials in $\dot{B}(t)$'s are to be defined. To have non linear functions of degree n, we must have $K^{-n}(R^n)$, the symmetric tensor product of $K^{-1}(R^1)$.

For instance, we discuss his idea of defining $(\frac{dB}{dt})^2$. Let us remind the simplest case of the Itô formular $(dB(t))^2 = dt$. This implies $(\frac{dB}{dt})^2 = \frac{1}{dt}$, which has no meaning. However, $(dB(t))^2$ is random, although it is infinitesimal. To define the square of $\dot{B}(t)$ there need some modification to overcome the difficulties above. Hida's idea is to apply the *renormalization* to the centered random quantity $(dB(t))^2 - dt$. Namely, magnify $(dB(t))^2 - dt$ by multiplication as much as $(\frac{1}{dt})^2$. Formaly writing,

$$:\dot{B}(t)^2:= \dot{B}(t)^2 - \frac{1}{dt}$$

The resulting triple

$$(L^2)^+ \subset L^2(\mathcal{S}^*, \mu) \subset (L^2)^-$$

is an infinite dimensional analogue of the Gel'fand triple

$$\mathcal{S}(R^d) \subset L^2(R^d) \subset \mathcal{S}^*(R^d).$$

Hida used the $\mathcal{T}$-transform

$$\mathcal{T}\varphi(\xi) = \int_{\mathcal{S}^*} e^{i\langle x,\xi\rangle}\varphi(x)\, d\mu(x)$$

in his Carleton University lecture notes as a fundamental tool to identify Brownian functionals. This transform has its root τ-transform in his 1970 Princeton University Press book. Under this transform, the white noise $\dot{B}(t)$ being regarded as a generalized Brownian functional corresponds to the function

$$i\xi(t) \exp[-2^{-1}\int_R \xi(u)^2\, du].$$

The $\mathcal{T}$-transform is Hida's vision to interpret Lévy's functional analyis. A related S-transform

$$S\varphi(\xi) = \int_{\mathcal{S}^*} \varphi(x+\xi)\, d\mu(x)$$

was later given by Kubo and Takenaka in 1980. For example, the function

$$F(\xi) = \int_0^1 \rho(t)\xi(t)^2 dt$$

mentioned above in Lévy's work is the S-transform of the generalized Brownian functional

$$\int_0^1 \rho(t) : \dot{B}(t)^2 : dt.$$

It was Hida's ingenious idea to use the $\mathcal{T}$-transform (equivalently the S-transform) as a tool to define and study generalized Brownian functionals.

White noise has long been regarded as an informal, mysterious, and imaginary object. Owe to the tremendous effort of Hida, there is a mathematical theory of white noise. We see many applications of this theory to Feynman integrals, Dirichlet forms, quantum probability, random fields, calculation of variation, infinite dimensional harmonic analysis, infinite dimensional rotation groups, stochastic partial differential equations, biology, to name just a few. And twenty-five years after 1975, white noise theory is listed as 60H40 under stochastic analysis in the MSC 2000 subject classifications.

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