On the classical formal group laws related to Euler's elliptic integral

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The contents of this paper are as follows.

§1. Introduction

The theory of formal group laws has several origins in mathematics. For example [2], the Rubin-Tate group and Witt group showed up in the class field theory. The principal logarithm attached to a one-dimensional commutative formal group law can be used by Ochanine [4] as the generating function of an elliptic genus in cohomological cobordism theory.

Though the concept of a set-theoretical groups has appeared in a history of mathematics earlier than that of formal group laws, we can find a naive formal group laws in Euler's elliptic integral. In 1756, Euler [251] obtained an addition theorem on elliptic integrals of first kind, this implicitly gave an elliptic formal group law which will be defined in section two. Since a formal group law consists of a certain kind of formal power series with even-number variables, we may say that the theory of naive formal group laws is coordination-theoretical method. There exist some non-coordination functorial methods, these are bialgebras (due to Dieudonné) and Hopf algebras and group-schemes (due to Grothendieck), which are categorically equivalent to each other.

Each theory of course has both advantages and disadvantages, when one want to prove an existence theorem, it is convenient to carry out non-coordination functorial method, however the coordinate method is indispensable to write down an explicit group law.

§2. Elliptic formal group laws

Let K be a torsionfree commutative ring with identity element. A formal power series $F = F(X,Y)$ in $K[[X,Y]]$ is called a one dimensional formal group law over K if and only if $F(F(X,Y),Z)=F(X,F(Y,Z))$ and $F(X,0)=F(0,X)=X$. Since F is commutative, there exists the logarithm l_F attached to F such that

$$l_F(F(X,Y)) = l_F(X) + l_F(Y), \quad \left.\frac{d l_F(t)}{dt}\right|_{t=0} = 1.$$

In fact, it is known that

$$l_F(t) = \int_0^t \frac{1}{\left.\frac{\partial F}{\partial X}\right|_{(0,Y)}} dY.$$

Here, F is called an elliptic formal group law if and only if

$$l_F(t) = \sqrt{A} \int_0^t \frac{1}{\sqrt{A + Ct^2 + Et^4}}\, dt$$

, where $A = u^2$ for an unit u in K, and C, $0 \neq E \in K$.

Recall some basic examples in calculus as follows.

(i) $\mathbb{G}_m(X,Y) = X + Y + XY$, $\quad \dfrac{\partial \mathbb{G}_m}{\partial X} = 1 + Y$,

$$l_{\mathbb{G}_m}(t) = \int_0^t \frac{dt}{1+t} = \log(1+t) \quad .$$

(ii) $F_{sin}(X,Y) = X\sqrt{1-Y^2} + Y\sqrt{1-X^2}$, $\quad \dfrac{\partial F_{sin}}{\partial X} = \sqrt{1-Y^2} - \dfrac{YX}{\sqrt{1-X^2}}$,

$$l_{F_{sin}}(t) = \int_0^t \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) \; .$$

(iii) $F_{tan}(X,Y) = (X+Y)(1-XY)^{-1}$, $\quad \dfrac{\partial F_{tan}}{\partial X} = (1+Y^2)(1-XY)^{-2}$,

$$l_{F_{tan}}(t) = \int_0^t \frac{dt}{1+t^2} = \tan^{-1}(t) \; .$$

(iv) $F_{Lor}(X,Y) = (X+Y)(1+XY)^{-1}$ (; Lorentz boost), $\dfrac{\partial F_{Lor}}{\partial X} = (1-Y^2)(1+XY)^{-2}$,

$$l_{F_{Lor}}(t) = \int_0^t \frac{dt}{1-t^2} = \frac{1}{2} \log\left(\frac{1+t}{1-t}\right) .$$

Hence F_{tan} and F_{Lor} are degenerated elliptic formal group laws.

Let $S(t)=\sum_{0\leq j} \gamma_j t^j$ be a formal power series in $K[[t]]$. It is known that $S(t)$ is invertible in $K[[t]]$ if and only if the constant term $\gamma_0=S(0)$ of $S(t)$ is an unit in K. If A is a square element of an unit in K, then

$$R(t)=\sqrt{A+Ct^2+Et^4}=\sqrt{A}+\frac{C}{2\sqrt{A}}t^2+\frac{1}{2\sqrt{A}}\left(E-\frac{C^2}{4A}\right)t^4+\cdots$$

is an invertible power series in $K[[t]]$.

Let $\mathbb{G}_E$ be an elliptic formal group law over K. Now suppose that $R(X)^{-1}dX + R(Y)^{-1}dY=0$ after the model of Darboux. Since $\frac{d\ell(X)}{dX}=\frac{\sqrt{A}}{R(X)}$, one sees that $R(X)=\sqrt{A}\,\frac{dX}{d\ell}$, $R(Y)=-\sqrt{A}\,\frac{dY}{d\ell}$. From $A\left(\frac{dX}{d\ell}\right)^2=A+CX^2+EX^4$, $A\left(\frac{dY}{d\ell}\right)^2=A+CY^2+EY^4$, it follows that $\frac{d^2X}{d\ell^2}=\frac{CX+2EX^3}{A}$, $\frac{d^2Y}{d\ell^2}=\frac{CY+2EY^3}{A}$.

Hence we have

$$Y^2\left(\frac{dX}{d\ell}\right)^2-X^2\left(\frac{dY}{d\ell}\right)^2=\frac{1}{A}(-X^2+Y^2)(A-EX^2Y^2),$$

$$Y\frac{d^2X}{d\ell^2}-X\frac{d^2Y}{d\ell^2}=\frac{1}{A}(-X^2+Y^2)(-2EXY).$$

Consider a logarithmic differential as follows.

$$(A - EX^2Y^2)^{-1} \frac{d}{d\ell} (A - EX^2Y^2)$$

$$= (A - EX^2Y^2)^{-1}(-2EXY)\left(Y\frac{dX}{d\ell} + X\frac{dY}{d\ell}\right)$$

$$= (-X^2+Y^2)\frac{1}{A}\left\{Y^2\left(\frac{dX}{d\ell}\right)^2 - X^2\left(\frac{dY}{d\ell}\right)^2\right\}^{-1}(-2EXY)\left(Y\frac{dX}{d\ell} + X\frac{dY}{d\ell}\right)$$

$$= (YR(X) + XR(Y))^{-1}\frac{d}{d\ell}(YR(X) + XR(Y)) \quad .$$

Therefore $(A - EX^2Y^2)^{-1}(YR(X) + XR(Y))$ is constant.

Since $R(0) = \sqrt{A}$, we obtain that

$$G_{\mathcal{E}}(X, Y) = \sqrt{A}\,(A - EX^2Y^2)^{-1}(YR(X) + XR(Y)).$$

This discovery is due to Euler in 1766 ([3+5] P.311).

The contribution of Fagnano to lemniscatic elliptic integrals is discussed by Siegel ([3], 1-11). In 1718, by using a heuristic method, Fagnano found the doubling formula of lemniscate-arcs.

Recalling the following elementary equation ;

$$\frac{X\sqrt{(1+BY^2)(1-EY^2)}+Y\sqrt{(1+BX^2)(1-EX^2)}}{1+EBX^2Y^2} - \frac{X\sqrt{\frac{1-EY^2}{1+BY^2}}+Y\sqrt{\frac{1-EX^2}{1+BX^2}}}{1-XY\sqrt{\frac{1-EX^2}{1+BX^2}}\sqrt{\frac{1-EY^2}{1+BY^2}}}$$

$$= \frac{(B-1)XY\left\{X(1-EY^2)\sqrt{(1+BY^2)(1-EX^2)}+Y(1-EX^2)\sqrt{(1+BX^2)(1-EY^2)}\right\}}{(1+EBX^2Y^2)\left\{\sqrt{(1+BX^2)(1+BY^2)}-XY\sqrt{(1-EX^2)(1-EY^2)}\right\}},$$

we have

$$(1+EX^2Y^2)^{-1}\left\{X\sqrt{(1+Y^2)(1-EY^2)}+Y\sqrt{(1+X^2)(1-EX^2)}\right\}$$

$$=\left\{1-XY\sqrt{\frac{1-EX^2}{1+X^2}}\sqrt{\frac{1-EY^2}{1+Y^2}}\right\}^{-1}\left\{X\sqrt{\frac{1-EY^2}{1+Y^2}}+Y\sqrt{\frac{1-EX^2}{1+X^2}}\right\}.$$

Thus, if $R(t)=\sqrt{(1+t^2)(1-Et^2)}$, then $G_E(X,Y)=(1-PQ)^{-1}(P+Q)$, where $P=X\sqrt{\frac{1-EY^2}{1+Y^2}}$, $Q=Y\sqrt{\frac{1-EX^2}{1+X^2}}$ (, see [252] P.100).

It reminds us quasi Landen transformation.

§3. Transformation of coordinates

In this section, we recall the transformations of elliptic integrals between Weierstrass cubic form and Abel-Jacobi quartic form. In basic calculus, it is generally called an integration by substitution or a change of variables.

For $Y = R(X) = \sqrt{A + BX + CX^2 + DX^3 + EX^4}$, we consider an integral $\mathcal{J} = \int_{\alpha}^{X} \frac{1}{R(X)}\, dX$, where $(X-\alpha) \mid (R(X))^2$, and $(X-\alpha)^2 \nmid (R(X))^2$.

Apply the following transformation to $\mathcal{J}$;

$$\begin{cases} X = \dfrac{1}{W} + \alpha \\ Y = \dfrac{Z}{W^2} \end{cases}$$

X	α	X
W	$+\infty$	$\frac{1}{X-\alpha} = W$

$$dX = -W^{-2}\, dW$$

It follows that

$$Z^2 = (B + 2C\alpha + 3D\alpha^2 + 4E\alpha^3)W^3 + (C + 3D\alpha + 6E\alpha^2)W^2 + (D + 4E\alpha)W + E .$$

Now write by $Z = A(W) = \sqrt{A_3W^3 + A_2W^2 + A_1W + A_0}$

, where $A_3 = f'(\alpha)$, $A_2 = \dfrac{f''(\alpha)}{2!}$, $A_1 = \dfrac{f^{(3)}(\alpha)}{3!}$, $A_0 = \dfrac{f^{(4)}(\alpha)}{4!}$

, and $f(X) = A + BX + CX^2 + DX^3 + EX^4$.

Hence we have

$$\mathcal{I} = \int_{+\infty}^{\frac{1}{X-\gamma}} \frac{W^2}{A(W)} \cdot \frac{dW}{-W^2} = \int_{W}^{+\infty} \frac{1}{A(W)}\, dW .$$

Since $A_3W^3 + A_2W^2 + A_1W + A_0 = \frac{A_3}{4}\left\{4\left(W+\frac{A_2}{3A_3}\right)^3 - \frac{4(A_2^2-3A_1A_2)}{3A_3^2}\left(W+\frac{A_2}{3A_3}\right) - \frac{4(A_1A_2-9A_0A_3)}{9A_3^2}\right\}$, we have

$$\mathcal{I} = \frac{2}{\sqrt{A_3}} \int_{V}^{+\infty} \frac{1}{\sqrt{4V^3 - g_2V - g_3}}\, dV$$

, where $V = W + \frac{A_2}{3A_3}$, $g_2 = \frac{4(A_2^2-3A_1A_2)}{3A_3^2}$, $g_3 = \frac{4(A_1A_2-9A_0A_3)}{9A_3^2}$.

By applying the theory of Weierstrass $\wp$-functions, one can write as follows.

$$Y^2 = 4X^3 - g_2X - g_3 = 4(X-e_1)(X-e_2)(X-e_3)$$

, where $e_1+e_2+e_3 = 0$, $e_1e_2+e_2e_3+e_1e_3 = -\frac{g_2}{4}$, $e_1e_2e_3 = \frac{g_3}{4}$.

Next take the following transformation of coordinates;

$$\begin{cases} V = e_1 + \dfrac{e_1-e_3}{U^2} \\[2ex] Y = \dfrac{Z}{U^3}\sqrt{e_1-e_3} \end{cases} \qquad \begin{array}{c|cc} V & V & +\infty \\ \hline U & U & 0 \end{array}$$

$$dV = \frac{-2(e_1-e_3)}{U^3}\, dU .$$

Then it follows from $Z^2=(12e_1^2-g_2)U^4+3e_1(e_1-e_3)U^2+4(e_1-e_3)^2$

that $\mathcal{I} = \frac{2}{\sqrt{A_3}}\int_V^{+\infty} \frac{U^3}{\sqrt{U^6}\sqrt{4V^3-g_2V-g_3}} \cdot \frac{-2(e_1-e_3)}{U^3}\, dU$

$$= \frac{4\sqrt{e_1-e_3}}{\sqrt{A_3}}\int_0^U \frac{dU}{\sqrt{(12e_1^2-g_2)U^4+3e_1(e_1-e_3)U^2+4(e_1-e_3)^2}} .$$

Thus we obtain elliptic formal group law attached to an elliptic integral of first kind.

§4. Realization by formalization of a linear algebraic group

It is known due to Dieudonné that there exists a formal group law G^* which is called the formalization of a linear algebraic group G.

Let $\tilde{X}=\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$ be the corresponding matrix in the Lie algebra of $SO(2)$. Since $\{SO^*(2)(X,Y)\}^{\sim}$

$$= (I_2-\tilde{X})^{-1}(\tilde{X}+\tilde{Y})(I_2+\tilde{X}\tilde{Y})^{-1}(I_2-\tilde{X})$$

$$=(1+X^2)^{-1}(1-XY)^{-2}\begin{pmatrix} -X(X+Y) & X+Y \\ -X-Y & -X(X+Y)\end{pmatrix}\begin{pmatrix} 1-XY & -X(1-XY) \\ X(1-XY) & 1-XY\end{pmatrix}=\begin{pmatrix} 0 & \frac{X+Y}{1-XY} \\ \frac{-(X+Y)}{1-XY} & 0\end{pmatrix},$$

we have $F_{tan} \cong SO^*(2)$. In general, we can prove that there exist $(n^2-n)/2$ one-dimensional Dieudonné typical subgroup laws of $SO^*(n)$ which are isomorphic to F_{tan}. On the other hand, the three-dimensional formal symplectic group law $Sp^*(1)$ is defined as follows.

$$
\begin{aligned}
& Sp^*(1)\left((X_1,X_2,X_3),(Y_1,Y_2,Y_3)\right) \\
&= \Big(D^{-1}(X_1+Y_1+X_1Y_1^2+X_1^2Y_1+X_2X_3Y_1+X_1Y_2Y_3+X_2Y_3-X_3Y_2), \\
&\quad D^{-1}(X_2+Y_2+X_2Y_1^2+X_1^2Y_2+X_1Y_2Y_3+X_2X_3Y_2+2X_1Y_2-2X_2Y_1), \\
&\quad D^{-1}(X_3+Y_3+X_3Y_1^2+X_1^2Y_3+X_3Y_2Y_3+X_2X_3Y_3+2X_3Y_1-2X_1Y_3) \Big)
\end{aligned}
$$

, where $D = 1+2X_1Y_1+X_1^2Y_1^2+X_2X_3Y_2Y_3+X_2Y_3+X_3Y_2+X_1^2Y_2Y_3+X_2X_3Y_1^2$.

The above explicit group law can be deduced by $\{Sp^*(1)(\mathbb{X},\mathbb{Y})\}^{\sim} = (I_2-\tilde{\mathbb{X}})^{-1}(\tilde{\mathbb{X}}+\tilde{\mathbb{Y}})(I_2+\tilde{\mathbb{X}}\tilde{\mathbb{Y}})^{-1}(I_2-\tilde{\mathbb{X}})$, where $\mathbb{X}=(X_1,X_2,X_3)$ denotes a generic point of $Sp^*(1)$ and $\tilde{\mathbb{X}} = \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{pmatrix}$ in $Lie(Sp(1))$.

Therefore, the typical subgroup law $Sp^*(1)\big|(X_1,0,0)$ is isomorphic to F_{Lor}. It could be an interesting problem to construct a theory of extensions of formal group laws.

Appendix 1. From Husserl's viewpoint

One of the most important objectives of a history of mathematics, which has been expressed by Weil [6], is regarded as a strategy for mathematician. In this appendix, we would like to recall some viewpoints due to Edmund Husserl in "The Origin of Geometry", which was written as an appendix of "The Crisis of European Sciences and Transcendental Phenomenology" (1934-1938).

According to a biography, Edmund Husserl (1859-1938) studied mathematics and physics at the universities of Leipzig and Berlin, where he was deeply influenced by the mathematician Carl Weierstrass (1815-1897), before moving to the University of Vienna, where he completed his doctorate in mathematics in 1882. Following a brief period as Weierstrass' assistant and a term in the army, Husserl went back to Vienna to study philosophy with Franz Brentano from 1884 to 1886.

Part II (of CESTP)

Clarification of the Origin of the Modern Opposition between Physicalistic Objectivism and Transcendental Subjectivism

§ 15. Reflection on the method of our historical manner of investigation

The type of investigation that we must carry out, and which has already determined the style of our preparatory suggestions, is not that of a historical investigation in the usual sense. Our task is to make comprehensible the teleology in the historical becoming of philosophy, especially modern philosophy, and at the same time to achieve clarity about ourselves, who are the bearers of this teleology, who take part in carrying it out through our personal intentions. We are attempting to elicit and understand the unity running through all the (philosophical) projects of history that oppose one another and work together in their changing forms. In a constant critique, which always regards the total historical complex as a personal one, we are attempting ultimately to discern the historical task which we can acknowledge as the only one which is personally our own.

[The Origin of Geometry]

. . . In principle, then, a history of philosophy, a history of the particular sciences in the style of the usual factual history, can actually render nothing of their subject matter comprehensible. For a genuine history of philosophy, a genuine history of the particular sciences, is nothing other than the tracing of the historical meaning-structures given in the present, or their self-evidences, along the documented chain of historical back-references into the hidden dimension of the primal self-evidences which underlie them. Even the very problem here can be made understandable only through recourse to the historical a priori as the universal source of all conceivable problems of understanding. The problem of genuine historical explanation comes together, in the case of sciences, with " epistemological " grounding or clarification.

. . . In any case, we can now recognize from all this that historicism, which wishes to clarify the historical or epistemological essence of mathematics from the standpoint of the magical circumstances or other manners of apperception of a time-bound civilization, is mistaken in principle. For romantic spirits the mythical-magical elements of the historical and prehistorical aspects of mathematics may be particulary attractive ; but to cling to this merely historically factual aspect of mathematics is precisely to love oneself to a sort of romanticism and to overlook the genuine problems, the internal-historical problem, the epistemological problem.

Appendix 2.

[251]

Opera Omnia XX p.58-79.

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DE INTEGRATIONE AEQUATIONIS DIFFERENTIALIS

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

Commentatio 251 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1761, p. 37—57
Summarium ibidem p. 7—9

SUMMARIUM

In hac dissertatione et nonnullis sequentibus, quibus simile argumentum pertractatur, quasi novus plane campus in Analysi aperitur integralia diversarum formularum, quae per se omnem integrationis solertiam respuunt, inter se comparandi. Cum enim ope notae comparationis angulorum relatio inter binas variabiles x et y huic aequationi differentiali

$$\frac{m\,dx}{\sqrt{(1-xx)}} = \frac{n\,dy}{\sqrt{(1-yy)}}$$

conveniens algebraice exhiberi queat, etsi utraque formula per se algebraice integrari nequit, sed angulum seu arcum circularem exprimit, haec relatio ex eo tantum fonte petita videtur, quod angulorum datam et quidem rationalem rationem tenentium sinus algebraice inter se comparari possunt. Neque talis comparatio locum habere videtur, nisi ambae formulae sive per angulos sive per logarithmos integrari queant. Quoties quidem solutio cuiusquam problematis ad huiusmodi aequationem differentialem $X\,dx = Y\,dy$, in qua X sit functio ipsius x et Y ipsius y, tantum perducitur, ea, quia variabiles sunt a se invicem separatae, tanquam penitus absoluta spectari solet, cum ope quadraturae duarum curvarum, quarum alterius area per $\int X\,dx$, alterius per $\int Y\,dy$ exprimitur, construi posset. Verum si pro dato quovis valore ipsius x valor ipsius y conveniens assignari debeat, id utramque quadraturam involvere videtur, sine qua relatio inter x et y minime exhiberi queat. Multo magis igitur mirum videbitur, cum talis formulae $\frac{dz}{\sqrt{(1-z^4)}}$ integrale neque per angulos neque per logarithmos exprimi possit, quae quantitates transcendentes ad comparationem solae idoneae putantur, nihilominus pro aequatione differentiali proposita relationem inter x

et y algebraice exhiberi posse, ita ut linea curva, cuius arcus indefinite hac formula integrali $\int \frac{dz}{\sqrt{(1-z^4)}}$ exprimitur, pari proprietate ac circulus sit praedita, ut scilicet omnes eius arcus inter se comparari seu proposito in eo arcu quocunque alius arcus, qui ad eam datam teneat rationem, geometrice assignari queat. Vel, quod eodem redit, aequatio integralis aequationis differentialis propositae, quae veram relationem inter x et y exprimit, non solum non tale integrale involvet, sed adeo erit algebraica.

Atque hoc quidem non tantum pro casu quodam particulari, verum adeo integrale completum, quod quantitatem constantem arbitrariam complectitur, erit algebraicum. Neque vero talis admiranda integratio in ipsa tantum aequatione differentiali locum habet, sed simili omnino modo Cel. Auctor ostendit hanc aequationem differentialem multo latius patentem

$$\frac{m\,dx}{\sqrt{(A+Bx^2+Cx^4)}} = \frac{n\,dy}{\sqrt{(A+By^2+Cy^4)}}$$

per aequationem algebraicam complete integrari posse, si modo numeri m et n sint rationales; quin etiam eandem integrandi methodum ad hanc aequationem multo generaliorem extendit

$$\frac{m\,dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}} = \frac{n\,dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}},$$

ubi in denominatoribus radicalibus omnes potestates ipsarum x et y ad quartam usque occurrunt. Hinc suspicari liceret, etiamsi hae potestates altius ascenderent, integrationem tamen algebraicam adhuc locum esse habituram; sed praeterquam quod methodus Auctoris in ipsa potestate quarta terminatur, facile ostendi potest, in potestate certe sexta algebraicam integrationem in genere excludi. Si enim coefficientes ita accipiantur, ut radix quadrata extrahi queat, ex hoc solo casu $\frac{m\,dx}{1+x^3} = \frac{n\,dy}{1+y^3}$ evidens est relationem inter x et y nequaquam algebraice exprimi posse, cum utriusque formulae integrale tam angulum quam logarithmum involvat; anguli autem et logarithmi certe inter se algebraice comparari non patiuntur. Interim tamen peculiari modo integratio huius quoque aequationis

$$\frac{m\,dx}{\sqrt{(A+Bx^2+Cx^4+Dx^6)}} = \frac{n\,dy}{\sqrt{(A+By^2+Cy^4+Dy^6)}}$$

algebraice exhibetur, unde patet hanc dissertationem multo plures investigationes continere, quam titulus quidem prae se ferre videtur.

1. Cum primum occasione inventionum Ill. Comitis FAGNANI[1]) hanc aequationem essem contemplatus, eiusmodi quidem relationem algebraicam inter

1) G. C. FAGNANO (1682—1766), *Produzioni matematiche*, T. 2, Pesaro 1750; *Opere matematiche*, T. 2, Milano-Roma-Napoli 1911. A. K.

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THEOREMA

9. *Dico igitur huius aequationis differentialis*

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}$$

aequationem integralem completam esse

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)}.$$

DEMONSTRATIO

Posita enim hac aequatione eius differentiale erit

$$xdx + ydy + ccxy(xdy + ydx) = (xdy + ydx)\sqrt{(1-c^4)},$$

unde fit

$$dx(x + ccxyy - y\sqrt{(1-c^4)}) + dy(y + ccxxy - x\sqrt{(1-c^4)}) = 0.$$

Ex eadem vero aequatione resoluta colligitur

$$y = \frac{x\sqrt{(1-c^4)} + c\sqrt{(1-x^4)}}{1+ccxx} \quad \text{et} \quad x = \frac{y\sqrt{(1-c^4)} - c\sqrt{(1-y^4)}}{1+ccyy}.$$

Si enim ibi radicali $\sqrt{(1-x^4)}$ tribuitur signum $+$, hic radicali $\sqrt{(1-y^4)}$ signum $-$ tribui debet, ut posito $x=0$ utrinque idem valor prodeat $y=c$. Erit ergo

$$x + ccxyy - y\sqrt{(1-c^4)} = -c\sqrt{(1-y^4)},$$

$$y + ccxxy - x\sqrt{(1-c^4)} = \quad c\sqrt{(1-x^4)},$$

quibus valoribus in aequatione differentiali substitutis prodit

$$-cdx\sqrt{(1-y^4)} + cdy\sqrt{(1-x^4)} = 0$$

sive

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

Huius ergo aequationis differentialis integrale est

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)},$$

et quia constantem c ab arbitrio nostro pendentem continet, erit simul integrale completum. Q. E. D.

10. Si igitur habeatur haec aequatio $\frac{dx}{\sqrt{(1-x^4)}}=\frac{dy}{\sqrt{(1-y^4)}}$, valor integralis completus ipsius x est

$$x=\frac{y\sqrt{(1-c^4)}\pm c\sqrt{(1-y^4)}}{1+ccyy},$$

unde, si constans arbitraria c evanescat, fit $x=y$; sin autem ponatur $c=1$, habemus $x=\pm\frac{\sqrt{(1-y^4)}}{1+yy}=\sqrt{\frac{1-yy}{1+yy}}$, qui sunt ambo illi valores particulares iam supra exhibiti. Hinc eruuntur alii valores particulares prae caeteris simpliciores, sed qui ad imaginaria devolvuntur. Ita posito $c=\infty$ fit

$$x=\frac{\sqrt{-1}}{y}$$

et posito $cc=-1$ fit

$$x=\sqrt{\frac{yy+1}{yy-1}},$$

qui itidem aequationi propositae satisfaciunt.

11. Quo autem ratio huius integralis clarius perspiciatur, concipiatur curva AM (Fig. 1), cuius haec sit indoles, ut posita abscissa $AP=u$ sit arcus

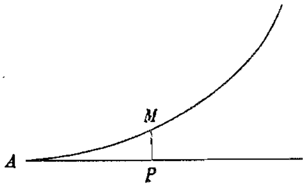

Fig. 1.

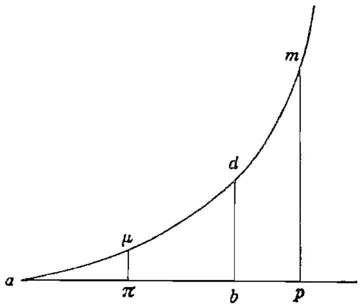

Fig. 2.

ei respondens $AM=\int\frac{du}{\sqrt{(1-u^4)}}$. Deinde eadem curva denuo (Fig. 2) descripta capiatur abscissa $ap=x$; erit arcus $am=\int\frac{dx}{\sqrt{(1-x^4)}}$. Sumto igitur

$$x=\frac{u\sqrt{(1-c^4)}\pm c\sqrt{(1-u^4)}}{1+ccuu}$$

fiet $\frac{dx}{\sqrt{(1-x^4)}}=\frac{du}{\sqrt{(1-u^4)}}$ ideoque arc. $am=$ arc. $AM+$ Const. Pro constantis autem huius determinatione posito $u=0$, quo casu arcus AM evanescit, fit $x=c$. Quare si capiatur abscissa $ab=c$, cui arcus ad respondeat, erit arcus $dm=$ arcui AM.

12. Ope huius ergo integrationis completae aequationis $\frac{dx}{\sqrt{(1-x^4)}} = \frac{du}{\sqrt{(1-u^4)}}$ in curva proposita arcui cuicunque AM, qui abscissae $AP=u$ respondet, arcus aequalis dm, qui a dato puncto d incipiat, abscindi poterit. Posita enim abscissa dato puncto d respondente $ab=c$ si capiatur abscissa

$$ap = x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu},$$

erit arcus dm arcui AM aequalis. Simili autem modo cum $\sqrt{(1-c^4)}$ negativum statui liceat, si capiatur abscissa

$$a\pi = \frac{c\sqrt{(1-u^4)} - u\sqrt{(1-c^4)}}{1+ccuu},$$

erit itidem arcus $d\mu$ arcui AM aequalis sicque in hac curva a dato quovis puncto d utrinque abscindi potest arcus dm et $d\mu$, qui arcui AM sint aequales.

13. Hinc ergo patet, si arcus ad aequalis capiatur arcui AM seu $c=u$, fore arcum am duplum arcus AM. Hinc si statuatur $ap = x = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$, prodibit arcus $am = 2$ arc. AM. Simili modo si capiatur arcus $ad = 2AM$ seu $c = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$ statuaturque $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu}$, obtinebitur arcus $am = 3$ arc. AM. Ac si iste valor ipsius x denuo pro c substituatur, ut sit $ad = 3AM$, iterumque statuatur $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu}$, nascetur arcus am quadruplus arcus AM; atque ita porro successive quaecunque multipla arcus AM geometrice assignari poterunt.

14. Sit arcus $ad = n \cdot AM$ et $ab = z$, ita ut sit

$$\int \frac{dz}{\sqrt{(1-z^4)}} = n \int \frac{du}{\sqrt{(1-u^4)}};$$

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Opera Omnia XX p. 80–107

OBSERVATIONES DE COMPARATIONE ARCUUM CURVARUM IRRECTIFICABILIUM

Commentatio 252 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1761, p. 58—84
Summarium ibidem p. 10—11

SUMMARIUM

Haec dissertatio ex eodem fonte est petita atque antecedens. Utraque enim innititur methodo formulas integrales, quae neque algebraice neque per angulos vel logarithmos expediri queant, algebraice inter se comparandi. Methodus autem ipsa, qua totum hoc negotium conficitur, ita est comparata, ut non data opera sit inventa, sed potius fortuito quasi detecta; ex quo, cum ad inventiones alias abstractissimas perduxerit, maxime digna videtur, ut omni studio uberius excolatur. In superiori quidem dissertatione hoc iam est praestitum, ut omnium curvarum, quarum arcus indefinite huiusmodi formula integrali $\int \frac{a\,dz}{\sqrt{(A + Bz + Cz^2 + Dz^3 + Ez^4)}}$ exprimuntur, arcus quicunque inter se comparari ac dato arcu quovis alii arcus ad eum datam rationem tenentes geometrice assignari queant, simili omnino modo, quo arcus circulares inter se comparari solent. Tali autem proprietate gaudet curva lemniscata vocari solita, cuius arcus indefinite hac formula $\int \frac{dz}{\sqrt{(1 - z^4)}}$ exprimitur, huiusque arcuum comparatio in hac dissertatione prolixius explicatur. Praeterea vero Cel. Auctor investigationes suas ad arcus ellipticos et hyperbolicos extendit, in quo nova omnino vis illius methodi cernitur, cum rectificatio ellipsis et hyperbolae nullo modo ad formulam integralem ante commemoratam revocari possit. Neque vero etiam in his curvis comparatio arcuum uti in circulo institui potest; sed, quod iam pridem in arcubus parabolicis est factum, id nunc etiam istius novae methodi beneficio in ellipsi et hyperbola praestatur. Scilicet dato in altera curva arcu quocunque a puncto etiam dato semper alius arcus in eadem curva abscindi potest, cuius ab illo differentiam geometrice assignare liceat; tum vero etiam negotium ita confici potest, ut non ipsorum arcuum, sed quorumvis eorum

THEOREMA 6

40. *Si corda arcus simplicis CM* (Fig. 13) *sit $= z$ et corda arcus n-cupli $CM^n = u$, erit corda arcus $(n+1)$-cupli*

$$CM^{n+1} = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - uz\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}}.$$

DEMONSTRATIO

Erit ergo ipse arcus simplex

$$CM = \int \frac{dz}{\sqrt{(1-z^4)}}$$

et arcus n-cuplus

$$CM^n = \int \frac{du}{\sqrt{(1-u^4)}} = n\int \frac{dz}{\sqrt{(1-z^4)}}$$

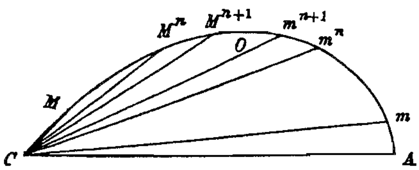

Fig. 13.

ideoque habemus $du = \frac{n\,dz\sqrt{(1-u^4)}}{\sqrt{(1-z^4)}}$. Ponamus brevitatis gratia

$$z\sqrt{\frac{1-uu}{1+uu}} = P \quad \text{et} \quad u\sqrt{\frac{1-zz}{1+zz}} = Q,$$

ut sit corda pro arcu $(n+1)$-cuplo exhibita $CM^{n+1} = \frac{P+Q}{1-PQ}$, quae dicatur $= s$, atque demonstrari oportet esse arcum huic cordae respondentem

$$\int \frac{ds}{\sqrt{(1-s^4)}} = (n+1)\int \frac{dz}{\sqrt{(1-z^4)}} \quad \text{seu} \quad \frac{ds}{\sqrt{(1-s^4)}} = \frac{(n+1)dz}{\sqrt{(1-z^4)}}.$$

Cum autem sit $s = \frac{P+Q}{1-PQ}$, erit

$$ds = \frac{dP(1+QQ) + dQ(1+PP)}{(1-PQ)^2};$$

tum vero reperitur

$$1 - s^4 = \frac{(1-PQ)^4 - (P+Q)^4}{(1-PQ)^4} = \frac{(1+PP+QQ+PPQQ)(1-PP-QQ-4PQ+PPQQ)}{(1-PQ)^4}$$

ergo

$$\sqrt{(1-s^4)} = \frac{\sqrt{(1+PP)(1+QQ)(1-PP-QQ-4PQ+PPQQ)}}{(1-PQ)^2},$$

ex quo elicitur

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP\sqrt{\frac{1+QQ}{1+PP}}+dQ\sqrt{\frac{1+PP}{1+QQ}}}{\sqrt{(1-PP-QQ-4PQ+PPQQ)}},$$

cuius expressionis ergo valorem investigemus.

Ac primo quidem est

$$1+PP=\frac{1+uu+zz-uuzz}{1+uu}\quad \text{et}\quad 1+QQ=\frac{1+uu+zz-uuzz}{1+zz},$$

ita ut sit $\frac{1+PP}{1+QQ}=\frac{1+zz}{1+uu}$ ideoque

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP\sqrt{\frac{1+uu}{1+zz}}+dQ\sqrt{\frac{1+zz}{1+uu}}}{\sqrt{(1-PP-QQ+PPQQ-4PQ)}}.$$

Deinde vero ob

$$1-PP=\frac{1+uu-zz+uuzz}{1+uu}\quad \text{et}\quad 1-QQ=\frac{1+zz-uu+uuzz}{1+zz}$$

erit

$$(1-PP)(1-QQ)=1-P^2-Q^2+P^2Q^2=\frac{1-z^4-u^4+4uuzz+u^4z^4}{(1+zz)(1+uu)}$$

et

$$4PQ=\frac{4uz\sqrt{(1-z^4)(1-u^4)}}{(1+zz)(1+uu)};$$

hincque concluditur denominator

$$\sqrt{(1-PP-QQ+PPQQ-4PQ)}$$
$$=\frac{\sqrt{(1-z^4-u^4+4uuzz+u^4z^4-4uz\sqrt{(1-z^4)(1-u^4)})}}{\sqrt{(1+zz)(1+uu)}}=\frac{\sqrt{(1-z^4)(1-u^4)}-2uz}{\sqrt{(1+zz)(1+uu)}},$$

ex quo obtinebitur

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP(1+uu)+dQ(1+zz)}{\sqrt{(1-z^4)(1-u^4)}-2uz}.$$

Iam vero differentiando elicimus

$$dP=dz\sqrt{\frac{1-uu}{1+uu}}-\frac{2zudu}{(1+uu)\sqrt{(1-u^4)}},$$
$$dQ=du\sqrt{\frac{1-zz}{1+zz}}-\frac{2zudz}{(1+zz)\sqrt{(1-z^4)}},$$

quare ob

$$du = \frac{n\,dz\,\sqrt{(1-u^4)}}{\sqrt{(1-z^4)}}$$

erit

$$dP = dz\sqrt{\frac{1-uu}{1+uu}} - \frac{2nuz\,dz}{(1+uu)\sqrt{(1-z^4)}},$$

$$dQ = \frac{n\,dz\sqrt{(1-u^4)}}{1+zz} - \frac{2uz\,dz}{(1+zz)\sqrt{(1-z^4)}},$$

unde conficitur numerator

$$dP(1+uu) + dQ(1+zz) = dz\sqrt{(1-u^4)} - \frac{2nuz\,dz}{\sqrt{(1-z^4)}} + n\,dz\sqrt{(1-u^4)} - \frac{2uz\,dz}{\sqrt{(1-z^4)}}$$

sive

$$dP(1+uu) + dQ(1+zz) = (n+1)dz\sqrt{(1-u^4)} - \frac{2(n+1)uz\,dz}{\sqrt{(1-z^4)}}$$

$$= \frac{(n+1)dz}{\sqrt{(1-z^4)}}\left(\sqrt{(1-z^4)(1-u^4)} - 2uz\right),$$

unde perspicuum est esse

$$\frac{ds}{\sqrt{(1-s^4)}} = \frac{(n+1)dz}{\sqrt{(1-z^4)}}$$

et

$$\text{arc. } CM^{n+1} = (n+1) \text{ arc. } CM.$$

Q. E. D.

COROLLARIUM 1

41. Si a vertice A abscindantur arcus Am, Am^n, Am^{n+1} arcubus CM, CM^n, CM^{n+1} respective aequales, erit Cm corda complementi arcus CM, Cm^n corda complementi arcus CM^n, Cm^{n+1} corda complementi arcus CM^{n+1}. Erunt autem ob cordas $CM = z$, $CM^n = u$, $CM^{n+1} = s$ complementorum cordae

$$Cm = \sqrt{\frac{1-zz}{1+zz}}, \quad Cm^n = \sqrt{\frac{1-uu}{1+uu}}, \quad Cm^{n+1} = \sqrt{\frac{1-ss}{1+ss}}.$$

Cum autem sit

$$s = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - zu\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}} = \frac{P+Q}{1-PQ},$$

erit

$$\sqrt{\frac{1-ss}{1+ss}} = \sqrt{\frac{1-PP-QQ-4PQ+PPQQ}{(1+PP)(1+QQ)}} = \frac{\sqrt{(1-z^4)(1-u^4)} - 2uz}{1+uu+zz-uuzz},$$

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Opera Omnia XX p. 302-317.

INTEGRATIO AEQUATIONIS

$$\frac{dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}}=\frac{dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}}$$

Commentatio 345 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae **12** (1766/7), 1768, p. 3–16
Summarium ibidem p. 5—6

SUMMARIUM

Calculus integralis, ad tantam hodie summorum Geometrarum studio perfectionem evectus, insignibus incrementis et subsidiis nunquam non ditatus fuit, quando ii aequationes differentiales solutu difficiliores, quarum integralia casu quasi vel per ambages et indirecte invenire ipsis licuerat, data opera meditationi subiecerunt methodos scrutaturi directas ad eadem, de quibus aliunde iam constitit, integralia perveniendi. Aequationis propositae integrale idque algebraicum et completum via admodum obliqua, cum in corporis ad duo centra virium fixa attracti motum inquireret, Ill. EULERO invenire licuit, qua is excitatus occasione istam integrationem data opera est aggressus eamque suis meditationibus eo censuit digniorem, quo plura et praeclariora Analyseos artificia difficultatum, quibus ea implicari videtur, evolutio, cum neutram partem seorsim ne ad arcus quidem circulares vel logarithmos revocare liceat, polliceri merito videbatur. En igitur directam methodum eamque substitutionibus et subsidiis analyticis notatu maxime dignis fundatam, qua propositae aequationis integrale eruitur cum priori perfecte congruens; quae cum sublatis difficultatibus potioribus dubium non sit, quin excoli possit uberius et ad brevitatem magis concinnam reduci, ad promovendos Analyseos fines plurimum momenti continere merito est censenda.

1. Methodo admodum singulari atque obliqua perveneram olim[1]) ad integrationem huius aequationis, cuius integrale idque adeo completum aequatione

1) L. EULERI Commentationes 251 et 261 (indicis ENESTROEMIANI); vide p. 58 et 153. A. K.

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ubi constans G ita accipi debet, ut formula irrationalis

$$\sqrt{AE(AE+CG+GG)}$$

non fiat imaginaria.

15. Forma haec integralis adhuc commodior reddi potest ponendo $G=Eff$ sicque fiet aequatio integralis

$$A(xx+yy)=ff(A+Exxyy)+2xy\sqrt{A(A+Cff+Ef^4)},$$

ubi f est constans arbitraria. Hinc autem elicitur

$$y=\frac{x\sqrt{A(A+Cff+Ef^4)}\pm f\sqrt{A(A+Cxx+Ex^4)}}{A-Effxx}$$

similique modo

$$x=\frac{y\sqrt{A(A+Cff+Ef^4)}\pm f\sqrt{A(A+Cyy+Ey^4)}}{A-Effyy}.$$

Quae formulae cum iis, quas olim[1]) dederam, perfecte consentiunt.

16. Integrale hic quidem aequationis differentialis propositae methodo directa sum consecutus, verumtamen diffiteri non possum hoc per multas ambages esse praestitum, ita ut vix sit expectandum cuiquam has operationes in mentem venire potuisse. Ex quo haec ipsa methodus, qua hic sum usus, plurimum in recessu habere videtur neque ullum est dubium, quin eam diligentius scrutando aditus ad multa alia praeclara aperiatur ac fortasse alia nova methodus idem praestandi detegatur, unde non contemnenda subsidia ad Analysin perficiendam hauriri queant.

17. Operationes hic adhibitae aliquantum variari possunt, quod probe perpendisse usu non carebit. Propositam scilicet aequationem differentialem ita refero

$$\frac{ydx}{xdy}=\sqrt{\frac{Ayy+Cxxyy+Ex^4yy}{Axx+Cxxyy+Exxy^4}}=\sqrt{\frac{P+Q}{P-Q}},$$

ut sit

$$\frac{P}{Q}=\sqrt{\frac{(A+Exxyy)(xx+yy)+2Cxxyy}{(A-Exxyy)(yy-xx)}},$$

1) L. EULERI Commentatio 261 (indicis ENESTROEMIANI); vide p. 153. A. K.

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